① changing of variable
② changing of order

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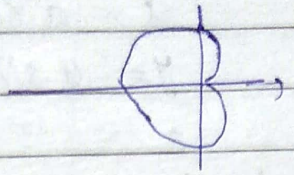
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Unit - 3

Multiple integrals

(a) $r = a(1 - \cos\theta)$ →

(b) $r = a(1 + \cos\theta)$



→ odd ⇒ n loop

$r = a\cos n\theta$

$r = a\sin n\theta$

→ n even then double of n loop

No. of loops in the curve

$r = a\cos n\theta$

or $r = a\sin n\theta$

(i) if n is odd then no. of loop is n

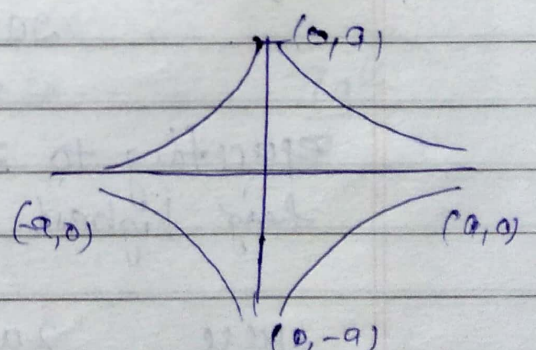
(ii) if n is even then no. of loop = 2n

$r^2 = a^2\cos 2\theta$

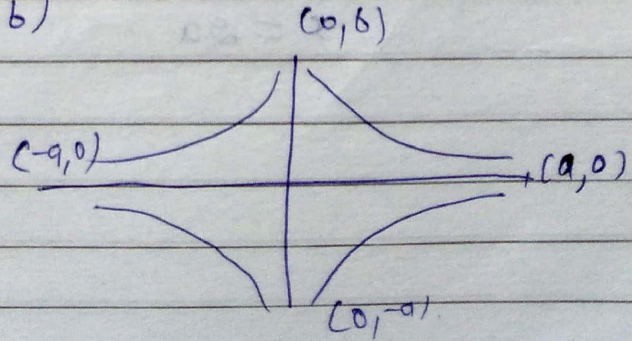
for asymptote x and y largest degree term coefficient is = 0.

~~X~~ Asternid :

(a) $x^{2/3} + y^{2/3} = a^{2/3}$



(b) $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$



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Parametric equation of Asteroid

$$x = a\cos^3\theta$$
$$y = a\sin^3\theta$$

Que: Find the area bounded by the curve $y^2(2a-x) = x^3$ and its asymptote

Sol: Origin: it is passes through origin

Symmetric: Replacing by $y = -y$ no change therefore Sym. about x axis.

tangent at origin:

$$2ay^2 = 0$$
$$y^2 = 0 \quad y = 0, 0$$

- $y^2 =$ sym. about x axis
  $x^2 =$ " " y axis

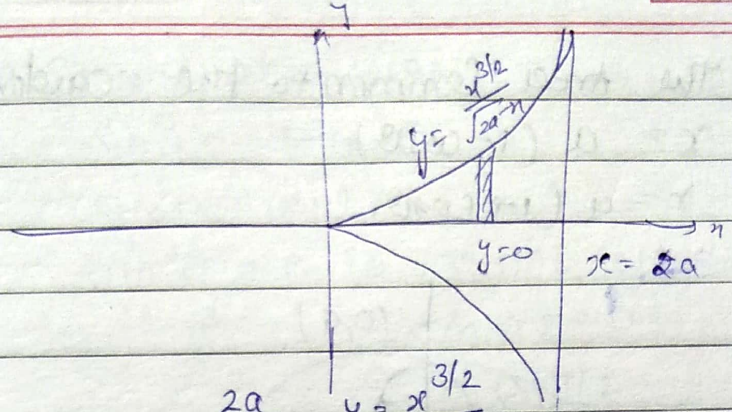
Asymptote:

$$2a - x = 0$$
$$x = 2a$$

equating to zero the coefficient of highest degree x and y.

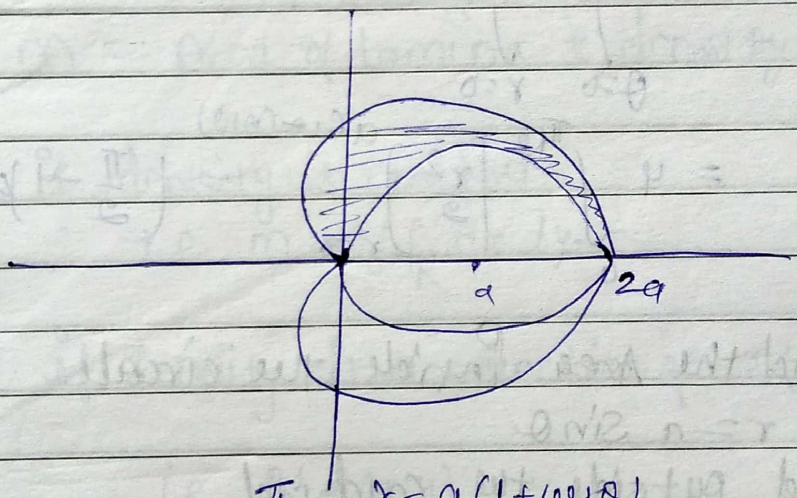
Here $2a - x = 0$

$x = 2a$



$$A = 2\int_{x=0}^{2a}\int_{y=0}^{y=\frac{x^{3/2}}{\sqrt{2a-x}}} dy \cdot dx$$

Q. find the area inside the cardioid $r = a(1+\cos\theta)$ and outside the circle $r = 2a\cos\theta$



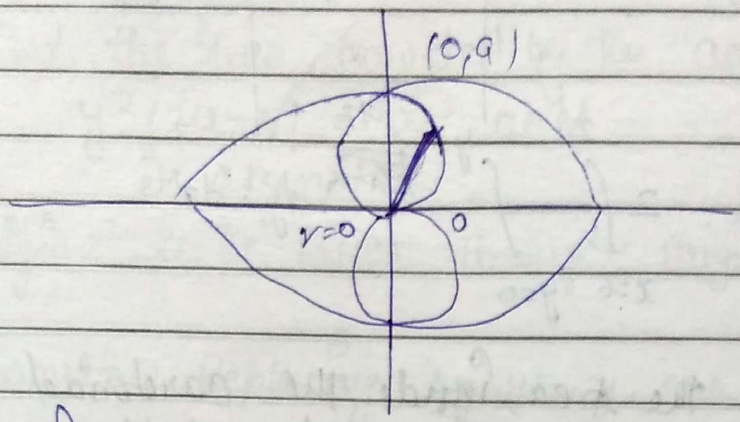
$$A = 2\int_{\theta=0}^{\pi}\int_{r=2a}^{r=a(1+\cos\theta)} r\,dr\,d\theta = \frac{\pi a^2}{2}$$

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Ques Find the area common to the cardioid

$r = a(1-\cos\theta)$

$r = a(1+\cos\theta)$



$A =$

$$A = 4\int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{a(1-\cos\theta)} r\,dr\,d\theta$$

$$= 4\int_{0}^{\pi/2}\left[\frac{r^2}{2}\right]_{r=0}^{a(1-\cos\theta)} = \left(\frac{\pi}{2}-4\right)a^2 \text{ Ans}$$

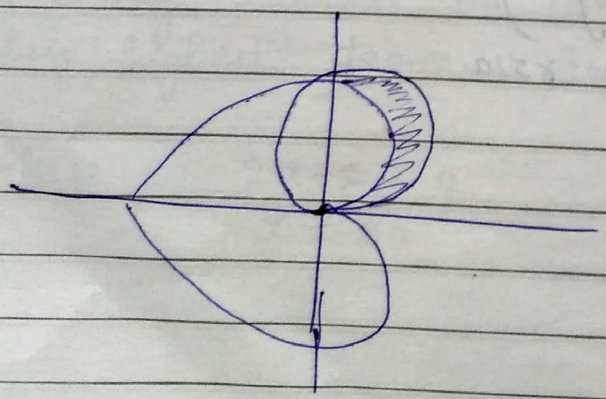
Ques Find the area inside the circle

$r = a\sin\theta$

and outside the cardioid

$r = a(1-\cos\theta)$

Soln



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$a \sin\theta = a(1-\cos\theta)$

$\sin\theta = 1-\cos\theta$

$\sin\theta + \cos\theta = 1$

$\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = 1$

$\sin 2\theta = 0$

$2\theta = 0, \pi/2$

$\theta = 0, \pi/2$

$$A = \int_{\theta=0}^{\pi/2} \int_{r=a(1-\cos\theta}^{r=a\sin\theta} r\,dr\,d\theta = \frac{a^2(4-\pi)}{4}$$

* Mass by double integral :

m = Area of Lamina x density

a) if density is constant

i.e $m = \iint_R dx\,dy$

b) if density is a function of x, y

i.e $\rho = f(x,y)$

$m = \iint_R f(x,y)\,dx\,dy$

$$m = \iint_R f(x,y)\cdot dx\,dy.$$

* Centre of Gravity :

Let $(\bar{x}, \bar{y})$

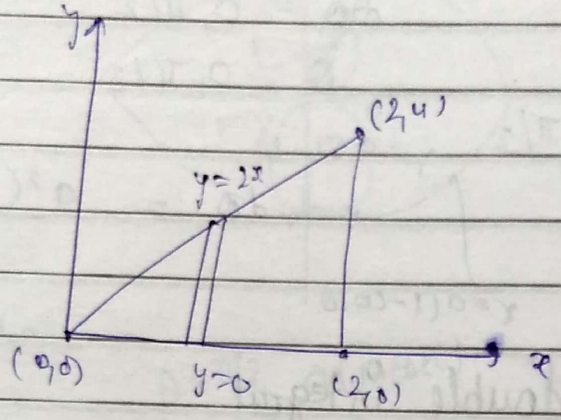
where $\bar{x} = \dfrac{\iint_R x\,dx\,dy}{\iint_R dx\,dy}$ $\quad \bar{y} = \dfrac{\iint_R y\,dx\,dy}{\iint_R dx\,dy}$.

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Que: A triangular plate with velocity (0,0) (2,0) (2,4) has density $\rho = 1+x+y$ then find

a) the mass of the plate

b) the position of its centre of gravity.



$$m = \iint \rho \, dx\,dy$$

$$= \int_0^2 \int_{y=0}^{2x} (1+x+y)\,dx\,dy$$

$$= \int_0^2 \left(y + xy + \frac{y^2}{2}\right)_0^{2x} dx$$

$$= \int_0^2 (2x + 2x^2 + 2x^2)\,dx$$

$$= \int_0^2 (2x + 4x^2)\,dx = \left(x^2 + \frac{4x^3}{3}\right)_0^2$$

$$= 4 + \frac{32}{3} = \frac{44}{3}$$

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$(C.G) = (\bar{x}, \bar{y})$

$$\bar{x} = \frac{\iiint \rho x\,dx\,dy}{\iiint \rho\,dx\,dy} \quad ; \quad \bar{y} = \frac{\iiint \rho y\,dx\,dy}{\iiint \rho\,dx\,dy}$$

i.e $\bar{x} = \frac{1}{m}\iiint \rho x\,dx\,dy \qquad \bar{y} = \frac{1}{m}\iiint \rho y\,dx\,dy$

$$\bar{x} = \frac{3}{44}\int_0^2\int_0^{2x} (1+x+y)\,x\,dx\,dy$$

$$\bar{x} = \frac{3}{44}\int_0^2 \left(xy + x^2y + \frac{xy^2}{2}\right)_0^{2x}$$

$$\bar{x} = \frac{3}{44}\int_0^2 \left(2x^2 + x^2\cdot 2x + \frac{4x^2}{2}\cdot x\right)dx$$

$$\frac{3}{44}\int_0^2 \left(2x^2 + 2x^3 + 2x^3\right)dx$$

$$= \frac{3}{44}\left[\frac{2x^3}{3} + \frac{2x^4}{4} + \frac{2x^4}{4}\right]_0^2$$

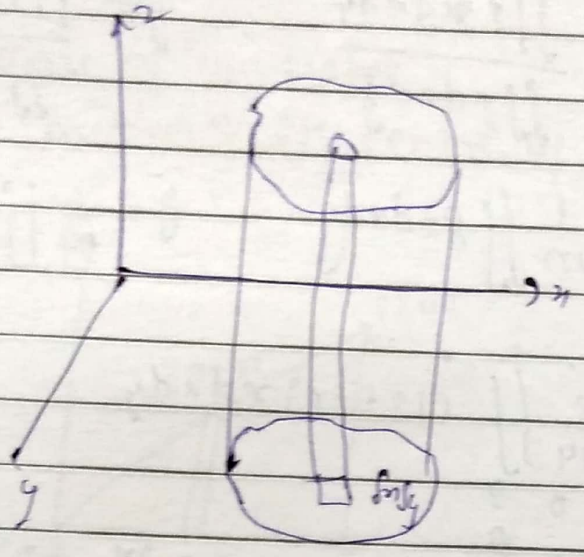
$$= \frac{3}{44}\left[\frac{2x^3}{3} + x^4\right]_0^2 = \frac{3}{44}\left(\frac{16+48}{3}\right)$$

$$= \frac{16}{11} \text{ Ans}$$

$$\bar{y} = \frac{1}{m}\iint \rho y\,dx\,dy = \frac{8}{11}$$

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* Volume by double integrals :-



Volume = Area of the base x height

Volume = $z\,dx\,dy$.

$$\text{Volume} = \iint_R z\,dx\,dy.$$

** Volume by double integral by revolution about any axis.

$$V = \iint 2\pi y\,dx\,dy.$$

Que: find the volume bounded by x, y plane, the cylinder $x^2+y^2=1$ and the plane $x+y+z=3$

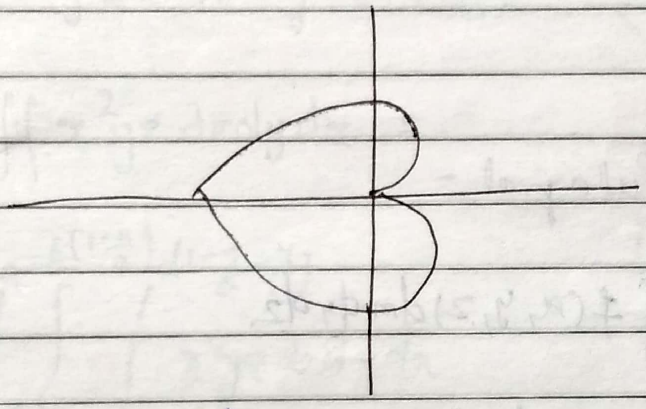
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$$V = \iint z \, dx \, dy$$

$$= \int_{y=\sqrt{1-x^2}} \int (3 - x - y) \, dx \, dy$$

$$= 3\pi - \frac{4}{3} \quad \text{Ans}$$

Que: Calculate by double integral the volume generated by the revolution of the cardioid $r = a(1 - \cos\theta)$ about its axis.



$$V = \iint 2\pi y \, dx \, dy$$

$$V = \iint 2\pi r \sin\theta \, |J| \, dr \, d\theta$$

$$V = \int_{\theta=0}^{\pi} \int_{r=0}^{r=a(1-\cos\theta)} 2\pi r^2 \sin\theta \, dr \, d\theta$$

$$r = \int_0^{\pi} \left(2\pi \sin\theta \frac{r^3}{3} \right)_0^{a(1-\cos\theta)} d\theta$$

$$= \int_0^{\pi} \left(2\pi \sin\theta \, a^3 \frac{(1-\cos\theta)^3}{3} \right)_0^{\pi}$$

$$= \int_0^{\pi} 2\pi \sin\theta \, a^3 \cdot \frac{(1 + \cos^3\theta + 3\cos\theta)}{3}$$

$$= \frac{8}{3}\pi a^3$$

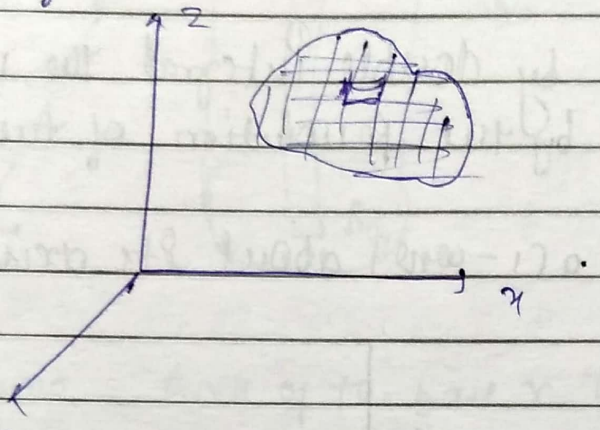
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* Triple Integration :

Consider a function

f

$$\text{Sum} = \sum_{r=1}^{n} f(x_r, y_r, z_r)\, \delta x, \delta y, \delta z$$



Triple integral =

$$= \iiint f(x, y, z)\, dx\, dy\, dz$$

$$\text{Vol.} = \int_{x_1=a}^{x_2=b} \int_{y_1=\phi(x)}^{y_2=\phi(x)} \int_{z_1=f_1(x,y)}^{z_2=f_2(x,y)} f(x, y, z)\, dx\, dy\, dz$$

volume by triple Integrals

$$V = \iiint dx\, dy\, dz$$

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Qu Evaluate $\iiint x^2yz\,dx\,dy\,dz$ through the Volume bounded by the Plane $x=0, y=0, z=0$ & $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

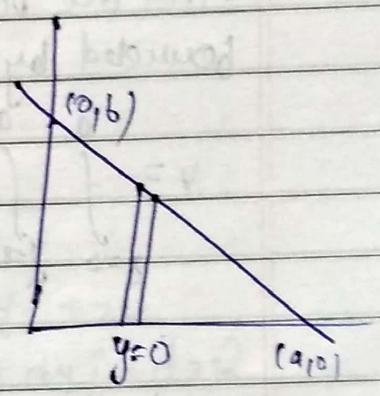
Sol^n Limits are

$z=0$ to $z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

Now in xy-plane $=0$

$x=0$, $y=0$ and $\frac{x}{a}+\frac{y}{b}=0$

$y=0$ to $y=b\left(1-\frac{x}{a}\right)$



$\Rightarrow \iiint x^2yz\,dx\,dy\,dz =$

$$\Rightarrow \int_{x=0}^{a}\int_{y=0}^{b\left(1-\frac{x}{a}\right)}\int_{z=0}^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} x^2yz\,dz\,dy\,dx$$

$$= \frac{1}{2}\int_{x=0}^{a}\int_{y=0}^{b\left(1-\frac{x}{a}\right)} x^2y\left(z^2\right)_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)}$$

$$= \frac{1}{2}\int_{x=0}^{a}\int_{y=0}^{b\left(1-\frac{x}{a}\right)} x^2yc^2\left(1-\frac{x}{a}-\frac{y}{b}\right)^2 dy\,dx$$

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$$\frac{1}{2}\int_{x=0}^{a}\int_{y=0}^{b(1-\frac{x}{a})}\int_{0}^{c(1-\frac{x}{a}-\frac{y}{b})} x^2 y z^2 \, dz\, dy\, dx$$

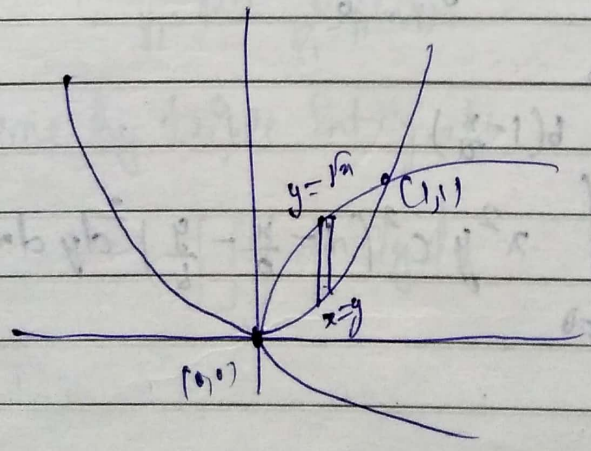
$$= \frac{1}{2}\int_{x=0}^{a}\int_{y=1}^{b(1-\frac{x}{a})} x^2 y c^2 \left(1-\frac{x}{a}-\frac{y}{b}\right)^2 dy\, dx$$

$$= \frac{a^2 b^2 c^2}{2520}$$

9. Find the volume of the tetrahedron bounded by the plane $x=0, y=0, z=0$

$$V = \int_{x=0}^{a}\int_{y=0}^{a-x}\int_{z=0}^{a-x-y} dx\, dy\, dz = \frac{a^3}{6}$$ Ans.

10. Find the volume of the region bounded by the surface $y=x^2$, $x=y^2$ and the plane $z=0$ to $z=3$



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$$V = \iiint dx\,dy\,dz$$

$$V = \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} \int_{z=0}^{3} = \int_{z=0}^{3} \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} dy\,dx\,dz = 1$$

Qw find the volume cutt of from the Paraboloac

$x^2 + \frac{y^2}{4} + z = 1$ by the Plane $z = 0$

Ans $= \pi$

Q.4 Vector Integration :

(1) Line Integration :

Position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

Consider a function $\vec{f}(\vec{r})$

$$\Sigma \vec{f}(r) \cdot \frac{d\vec{r}}{dt}$$

this summation is known as the line of integration along the curve C and it is denoted by

$$\int_C \vec{f} \cdot \frac{d\vec{r}}{dt} \cdot dt = \int_C \vec{F} \cdot d\vec{r}$$

If C is closed. then it is denoted as

$$\oint \cdot \vec{F} \cdot d\vec{r}$$

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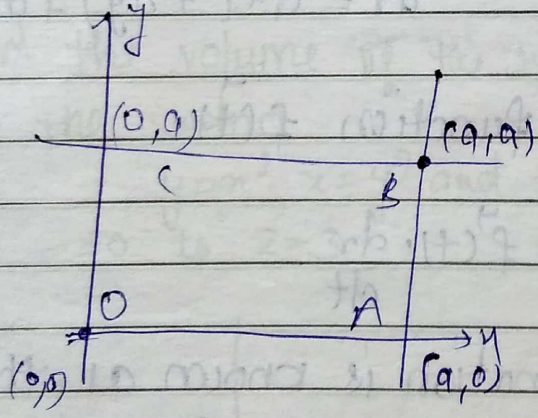
Note: If the position vector $\vec{r}$ displayed from the point A to B by the force $\vec{F}$ then total work done

$$= \int_A^B \vec{F} \cdot d\vec{r}$$

Qu: Evaluate

$$\int_C \vec{f} \cdot d\vec{r}$$ when

$\vec{F} = x^2\hat{i} + xy\hat{j}$ and c is bounded of the square in the plane $z=0$ and bound at by the lines $x=0, y=0, x=a$ and $y=a$.



$$\int_C \vec{F} \cdot d\vec{r} \Rightarrow \int_{OA} \vec{F} \cdot d\vec{r} + \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CO} \vec{F} \cdot d\vec{r}$$

(i) along OA, x varies from $x=0$ to $x=a$ and $y=0$ $dy=0$

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$$\int_{OA} \vec{F} \cdot d\vec{r} = \int_0^a (x^2\hat{i} + 0) \cdot (dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_0^a x^2\,\hat{i} + dx\,\hat{i} =$$

$$= \left(\frac{x^3}{3}\right)_0^a = \frac{a^3}{3} \quad \text{Ans}$$

(ii) Along AB $x = a$, $dx = 0$

$y = 0$ to $y = a$

$$\int_{AB} \vec{F} d\vec{r} = \int_{y=0}^{a} (a^2\hat{i} + ay\hat{j})(0\hat{i} + dy\hat{j}).$$

$$= \int_0^a dy\,dy = a\left(\frac{y^2}{2}\right)_0^a = \frac{a^3}{2} \quad \text{Ans}$$

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Q. If $\vec{F} = (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$, evaluate the line integral $\int \vec{A}\cdot d\vec{r}$ from (0,0,0) to (1,1,1) along the curve C as $x=t,\ y=t^2,\ z=t^3$

Sol^n

$r = x\hat{i} + y\hat{j} + z\hat{k}$

$dr = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$A\cdot dr = (3x^2+6y)dx - 14yz\,dy + 20xz^2\,dz$

from O(0,0,0) to A(1,1,1)

we get $t=0$ to $t=1$

$$\therefore \int_{OA} \vec{A}\cdot d\vec{r} = \int_0^1 \left(3t^2+6t^2\right)dt - 14t^2\cdot t^3\cdot 2t\,dt + 20t\cdot t^6\cdot 3t^2\,dt$$

$$= \int_{t=0}^{1} \left(9t^2dt + 28t^6 + 60t^9\right)dt$$

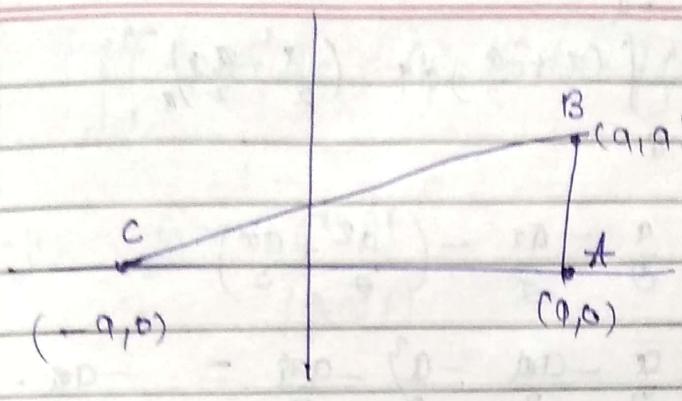
$$= \left(\frac{9t^3}{3} - \frac{28t^7}{7} + \frac{60t^{10}}{10}\right)_0^1$$

$$= 3 - 4 + 6 = 5 \quad \text{Ans}$$

Q. $F = y\hat{i} + x\hat{j}$

evaluate $\int_C \vec{F}\cdot d\vec{r}$, where C is a triangle (0,0) (a,0) (−a,0)

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$$\int_C \vec{F}\cdot d\vec{r} = \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CA} \vec{F}\cdot d\vec{r}$$

i) Along AB, y varies ~~from~~ a to 0

$$\int_{AB} \vec{F}\cdot d\vec{r} = \int_0^a a\,dy = a\cdot a = a^2$$

(ii) Along BC, equation of BC line

$$y - a = \frac{0-a}{-a-a}(x-a)$$

$$y - a = \frac{+a}{+2a}(x-a)$$

$$y - a = \frac{1}{2}(x-a)$$

$$y = \frac{x}{2} - \frac{a}{2} + a$$

$$y = \frac{x}{2} + \frac{a}{2} \Rightarrow \frac{x+a}{2}$$

$$\frac{dy}{1} = \frac{dx}{2}$$

$$\vec{F}\cdot d\vec{r} = \frac{1}{2}(x+a)dx = x\,\frac{dx}{2}$$

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int_a^{-a} \left(\frac{1}{2}x + \frac{a}{2} + \frac{x}{2}\right) dx \Rightarrow$$

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$$= \int_{a}^{-a} \left(x + \frac{a}{2}\right) dx = \left(\frac{x^2}{2} + \frac{ax}{2}\right)_{a}^{-a}$$

$$= \frac{a^2}{2} - \frac{ax}{2} - \left(\frac{a^2}{2} + \frac{ax}{2}\right)$$

$$= \frac{a^2}{2} - \frac{aa}{2} - \frac{a^2}{2} - \frac{ax}{2} = -aa = -a^2$$

* **Surface integration:**

Any integral which is evaluated over a surface is called the surface integral. Let $\vec{F}$ be a vector point function.

Let R be the Projection of the Surface S in xy plane and

Let, $\hat{n} = \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ be the direction Cosine of the $\hat{n}$.

$\therefore$ dxdy = Projection of ds in xy plane.

$dxdy = ds \cdot \cos\gamma$.

$$\cos\gamma = \frac{dxdy}{ds}$$

$$ds = \frac{dxdy}{|\hat{n}\cdot\hat{k}|}$$

$$\therefore \iint \vec{F}\cdot\hat{n}\, ds = \iint_R \vec{f}\cdot\hat{n}\, \frac{dxdy}{|\hat{n}\cdot\hat{k}|}$$

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$$\iint (\vec{F}\cdot\hat{n})\,ds = \iint \vec{F}\cdot\hat{n}\,ds = \iint \vec{F}\cdot d\vec{s}$$

Note: $\hat{n} = \dfrac{\text{grad } f}{|\text{grad } f|}$

Qu Evaluate

$$\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k})\cdot ds$$

where S is the surface of sphere $x^2+y^2+z^2 = a^2$ in the first octant

$$\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k})\cdot \hat{n}\,ds$$

where $\hat{n} = \dfrac{\text{grad } f}{|\text{grad } f|}$

$\because f = x^2+y^2+z^2 - a^2$

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$\nabla f = \hat{i}2x + \hat{j}2y + 2z\hat{k}$$

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{2(x\hat{i}+y\hat{j}+z\hat{k})}{\sqrt{4x^2+4y^2+4z^2}}$$

$$\frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+\cdots}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}$$

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Now, $\vec{F}\cdot\hat{n} = \frac{3xyz}{a}$

$$\iint_S (yz\hat{i} + xz\hat{j} + xy\hat{k})\cdot\hat{n}\, ds$$

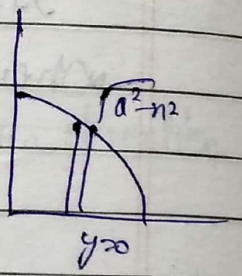
$$= \iint_S \frac{3}{a} xyz\, dx = \frac{3}{a}\iint_S xyz\, dx$$

$$= \frac{3}{a}\iint_R xyz\, dx\, dy$$

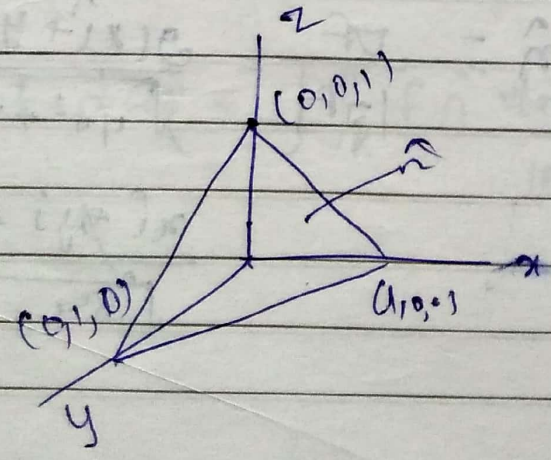
$$= \frac{3}{a}\iint_R xyz \frac{dx\, dy}{|\hat{n}\cdot\hat{k}|}$$



Here $\hat{n} = \hat{k}$



Qu: Evaluate $\iint_S \vec{A}\cdot\hat{n}\, ds$, where $\vec{A} = 12x^2y\hat{i} - 3yz\hat{j} + 2y\hat{k}$ and S is the portion of the plane $x+y+z=1$ included in the first octant.



Surface = volume
Gauss divergence Theorem

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Let $f = x + y + z + 1 = 0$

Find
(1) $\hat{n} = \frac{\nabla f}{|\nabla f|}$
(2) $\vec{F} \cdot \hat{n}$
(3) Projection $|\hat{n} \cdot \hat{k}|$

$\nabla f = \hat{i} + \hat{j} + \hat{k}$

$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{i + j + k}{\sqrt{3}}$

$\vec{A} \cdot \hat{n} = \frac{12x^2y - 3yz + 2y}{\sqrt{3}}$

Projection in xy plane, $z = 0$

$x + y = 1$

(4) $ds = \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$ ← Projection

$\therefore |\hat{n} \cdot \hat{k}| = \frac{1}{\sqrt{3}}$

$\therefore \iint_S \vec{A} \cdot \hat{n}\, ds = \iint_R (\vec{A} \cdot \hat{n}) \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$

$= \frac{1}{\sqrt{3}} \int_0^1 \int_{y=0}^{1-x} (12x^2y - 3yz + 2y) \frac{dx\,dy}{1/\sqrt{3}}$

$= \int_0^1 \int_{y=0}^{1-x} \{12x^2y - 3y(1 - x - y) + 2y\}\, dx\,dy$

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* Volume Integration :-

any integral which is to evaluated over a volume is called a volume integral if V is the volume bounded by the surface S then the triple integral

$$\iiint \phi \, dv \quad \text{(Scalar triple integral)}$$

and $\iiint \vec{F} \, dv$ (vector volume integrals)

$$\iiint (F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}) \, dx\,dy\,dz$$

Q. Evaluate $\iiint \phi \, dv$, where $\phi = 45x^2y$ and V is closed region bounded by the planes

$$4x + 2y + z = 8,\ x = 0,\ y = 0,\ z = 0$$

$$\iiint \phi \, dv = \iiint 45x^2y \, dx\,dy\,dz$$

$$\int_{x=0}^{2} \int_{y=0}^{\frac{8-4x}{2}} \int_{z=0}^{8-4x-2y} 45x^2y \, dx\,dy\,dz$$

Ans = 128

10 marks in any of one theorem

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Gauss divergence theorem :

If $\vec{F}$ is a vector point fn having cont. first order partial derivatives in the region V is bounded by a close surface S then

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \text{div}\, \vec{F}\, dV$$

where $\hat{n}$ is the unit normal vector in outward dir.

Q. Find

$$\iint_S \vec{F} \cdot \hat{n}\, ds, \text{ where } \vec{F} = (2x+3z)\hat{i} - (xz+y)\hat{j} + (y^2+2z)\hat{k}$$

and S is the surface of the sphere having center at (3, 1, -2) and Radius 3.

Soln: We have by Gauss div. theorem.

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \text{div}\, F\, dv$$

Now $\text{div}\, F = \nabla \cdot \vec{F}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left((2x+3z)\hat{i} - (xz+y)\hat{j} + (y^2+2z)\hat{k}\right)$$

$$\nabla \cdot F = 2 - 1 + 2 = 3$$

$$\text{div}\, \vec{F} = 3$$

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V 3\, dv$$

$$= 3V$$

$$= 3 \times \frac{4}{3}\pi (3)^3 = 4 \times 27\pi$$ Ans

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Que Verify div. theorem for $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taking over region bounded by the cylinder $x^2 + y^2 = 4$, $z = 0, z = 3$

We have gauss div. theorem.

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \text{div}\vec{F}\, dv$$

$$\text{div}\vec{F} = \vec{\nabla}\cdot\vec{F}$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot(4xi - 2y^2j + z^2k)$$

$$= 4 - 4y + 2z$$

$$\iiint \text{div}\vec{F}\, dv = \iiint (4 - 4y + 2z)\, dx\, dy\, dz$$

$$= \int_{z=0}^{3}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x=-2}^{2} (4 - 4y + 2z)\, dx\, dy\, dz$$

$$= \int_{z=0}^{3}\int_{x=-2}^{2}\left(4y - \frac{4y^2}{2} + 2yz\right)_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}$$

$$= 2\int_{z=0}^{3}\int_{x=-2}^{2}\int_{0}^{\sqrt{4-x^2}} (4 - 4y + 2z)\, dy\, dx\, dz$$

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$$= 2\int_{z=0}^{3}\int_{x=-2}^{2}\left[4y - \frac{4y^2}{2} + 2yz\right]_0^{\sqrt{4-x^2}}$$

$$= 2\int_0^3\int_{-2}^{2}\int_0^{\sqrt{4-x^2}} (4+2z)\,dy\,dx\,dz$$

$$= 2\int_0^3\int_{-2}^{2}(4+2z)\sqrt{4-x^2}\,dx\,dz$$

$$= 4\int_0^3\int_0^2 (4+2z)\sqrt{4-x^2}\,dx\,dz$$

$$= 4\int_0^3 (4+2z)\left\{\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right\}_0^2 dz$$

$$= 4\int_0^3 (4+2z)\{\pi\}$$

$$\therefore 4\pi\int_0^3 (4+2z) = 4\pi\left[4z + z^2\right]_0^3$$

$$= 4\pi[12+9]$$

$$= 84\pi$$ Ans

- S. along

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* Surface integral

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} \vec{F}\cdot\hat{n}\,ds + \iint_{S_2} \vec{F}\cdot\hat{n}\,ds + \iint_{S_3} \vec{F}\cdot\hat{n}\,ds$$

$$F = 4xz\hat{i} - 2y^2\hat{j} + z^2\hat{k}$$

① Surface S_1

$z=0$, $\hat{n} = -\hat{k}$

$$\vec{F}\cdot\hat{n} = -z^2 = 0$$

$$\therefore \iint_{S_1} \vec{F}\cdot\hat{n}\,ds = 0$$

Surface ② S_2

Let $f = x^2+y^2-4 = 0$

$$\nabla f = 2x\hat{i} + 2y\hat{j}$$

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4x^2+4y^2}}$$

$$\vec{F}\cdot\hat{n} = 4x\cdot 2x - 2y^2\cdot 2y$$

$$= 4x\cdot\frac{x}{2} - 2y^2\frac{y}{2} = 2x^2 - y^3$$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds =$$

Let R be the Projection in yz plane

So,

$$|\hat{n}\cdot\hat{i}| = \frac{x}{2}$$

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$$\iint_R (2x^2 - y^3)\frac{z}{x}\,dy\,dz$$

in yz plane

$x=0 \therefore y^2 = 4 \Rightarrow y = \pm 2$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \int_{-2}^{2}\int_{0}^{3}(2x^2 - y^3)\frac{z}{x}\,dz\,dy$$

$$= 2\int_{-2}^{2}\int_{0}^{3}\left(x - \frac{y^3}{x}\right)dy\,dz$$

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

$= r^2\cos^2\theta + r^2\sin^2\theta = 4$

$r^2 = 4$

$r = 2$

$x = 2\cos\theta$, $y = 2\sin\theta$, z

$$= \int_{\theta=0}^{\theta=2\pi}\int_{0}^{3}(2x^2 - y^3)2\,d\theta\,dz$$

$$= \int_{0}^{2\pi}\int_{0}^{3}(2\cdot 4\cos^2\theta - 8\sin^3\theta)2\,d\theta\,dz$$

$$= 16\int_{0}^{2\pi}\int_{0}^{3}(\cos^2\theta - \sin^3\theta)\,d\theta\,dz$$

$$= 8\times 16\int_{0}^{2\pi}(\cos^2\theta - \sin^3\theta)\,d\theta\,[z]_0^3\,d\theta$$

$$= 48 \int_0^{2\pi} (\cos^2\theta - \sin^3\theta)\, d\theta$$

$$= 48 \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx, \text{ if } f(2a-x) = f(x)$$

$$= 0, \text{ if } f(2a-x) = -f(x)$$

$$= 96 \int_0^{\pi} (\cos^2\theta)\, d\theta \longrightarrow$$

$$= 96 \times 2 \int_0^{\pi/2} \cos^2\theta\, d\theta =$$

$$= 96 \times 2 \left(\frac{1}{2}\right) \pi/2 = 48\pi \text{ Ans}$$

(iii) Surface integral over S_3 :

$$z = 3, \quad \hat{n} = \hat{k}$$

$$\vec{F} \cdot \hat{n} = z^2 = 9$$

$$\iint_{S_3} \vec{F} \cdot \hat{n}\, ds = \iint_S 9\, ds.$$

$$= 9 (\text{Area of the Surface})$$

$$= 9 \times \pi (2)^2$$

$$= 36\pi$$

Total Surface $= 0 + 48\pi + 36\pi = 84\pi$

Ans

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Qu:- Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ so that

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{8}{3}$$

Where $\vec{F}$ $4xz\hat{i} - y^2\hat{j} + yz\hat{k}$

and S is the surface of the cube bounded by the plane $x=0, x=1, y=0, y=1, z=0, z=1$

Sol^n:

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint \text{div}\,\vec{F}\,dv$$

$$\text{div}\,\vec{F} = \nabla\cdot\vec{F}$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(4xz\hat{i} - y^2\hat{j} + yz\hat{k}\right)$$

$$= 4z - 2y + y = 4z - y$$

$$\int_0^1\int_0^1\int_0^1 (4z - y)\,dx\,dy\,dz$$

$$\int_0^1\int_0^1 (4z - y)\,dy\,dz = \int_0^1 \left[4zy - y^2\right]_0^1 dz$$

$$= \int_0^1 (4z - 1)\,dz$$

$$= \left(4z^2 - z\right)_0^1$$

$$= 4 - 3 = (3)$$

SURFACE AREA

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2D, 3D both

(2) Stock's theorem :-

$$\oint_S \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\, \vec{F} \cdot \hat{n} \cdot ds$$

where $\hat{n}$ is a unit normal vector at any point of drawn in the sense in which a right handed screw would advanced when rotate in the sense of description of c.

only 2D * Green's theorem in XY Plane.

If c is a regular closed curve in xy Plane and S is a region bounded by c then

$$\oint_C (M dx + N dy) =$$

$$= \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy.$$

Gauss — Stock's — Green

Gauss → 4D

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(1) Gauss div. theorem:

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \operatorname{div} \vec{F}\, dv$$

(2) Stokes's theorem:

$$\oint_S f \cdot dr = \iint_S \operatorname{Curl} \vec{f} \cdot \hat{n}\, ds$$

(3) Green theorem in x-y plane

$$\oint_C (M dx + N dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$