Unit - 2

DATE: / 20
PAGE No

06/02/19

Linear Differential Equation of n^{th} order with Constant Coefficient:

* D.E of the form

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots P_n = R(x) \quad \text{--- (i)}$$

where $P_0, P_1, P_2, P_3 \cdots P_n$ are function of Constant and $R(x)$ is fn of x

If $R(x) = 0$
then eq^n (i) is known as the Complementry homogenious linear d.e of n^{th} order and its sol^n is known as complementry fn.

* Method to obtain complementry fn

Consider a second order linear d.e with constant coffficient

$$P_0 \frac{d^2 y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0$$

where P_0, P_1 and P_2 is constant --

DATE : / 20
PAGE No

Let $\frac{dy}{dx} = Dy$, $\frac{d^2y}{dx^2} = D^2y$

$$P_0 D^2 y + P_1 Dy + P_2 y = 0$$

$$(P_0 D^2 + P_1 D + P_2) y = 0 \quad -(ii)$$

$$f(D) y = 0$$

$$P_0 D^2 + P_1 D + P_2 = 0$$

Put $D = m$

$$P_0 m^2 + P_1 m + P_2 = 0 \rightarrow \text{Auxiliary eq}^n \quad -(iii)$$

eq^n (3) is known as the Auxiliary eq^n (AE)

Solve eq^n (3) and obtains the roots of eq^n (3)

(i) If roots of eq^n (iii) are real and distinct.
i.e m_1, m_2 then

$$\boxed{CF = C_1 e^{m_1 x} + C_2 e^{m_2 x}}$$

(ii) If roots of eq^n (3) are real and equal.

$$m_1 = m = m_2$$

$$\boxed{CF = C_1 e^{mx} + C_2 x e^{mx}}$$

(iii) If roots of eq^{n} (iii) are of the form

$$m = \alpha \pm i\beta$$

$$C.F = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$$

Qu Solve the D.E.

$$D^2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$$

Sol: $m^2 + 3m + 2y = 0$

$$m = \frac{-3 \pm \sqrt{9 - 8}}{2}$$

$$m = \frac{-3 \pm \sqrt{1}}{2}$$

$$m = -1, -2$$

$$C.F = C_1 e^{-1x} + C_2 e^{-2x}$$

Sol: $(D+1)(D+2)y = 0$

Let $(D+1)y = 0$

$$Dy + y = 0$$

$$\frac{dy}{dx} + y = 0 \qquad \int\frac{dy}{y} + \int dx = 0$$

DATE : / 20
PAGE No

* Rule/method to obtain the Particular Integral (PI) :

$$f(D)y = R(x)$$

$$y = \frac{R(x)}{f(D)}$$

$$PI = \frac{R(x)}{f(D)}$$

Case 1 If $R(x) = e^{ax}$.

$$P.I = \frac{e^{ax}}{f(D)} = \frac{e^{ax}}{f(a)}, \quad f(a) \neq 0$$

If $f(a) = 0$, then

$$P.I = \frac{x e^{ax}}{f'(D)} = \frac{x e^{ax}}{f'(a)}, \quad f'(a) \neq 0.$$

Q. $(D^2y + 4Dy + 4y) = 2\sin 2x$

For CF solution :

$$D^2y + 4Dy + 4y = 0$$

$$(D^2 + 4D + 4)y = 0$$

A.E is $(D = m)$

$$m^2 + 4m + 4 = 0$$

$$m = -2, -2$$

$$C.F = c_1 e^{-2x} + c_2 x e^{-2x}$$

DATE : / 20
PAGE No

$$P.I = \frac{R(x)}{f(D)}$$

$$= \frac{2\sinh 2x}{D^2+4D+4}$$

$$= \frac{2\left(\frac{e^{2x}-e^{-2x}}{2}\right)}{D^2+4D+4} = \frac{e^{2x}}{D^2+4D+4} - \frac{e^{-2x}}{D^2+4D+4}$$

$$= \frac{e^{2x}}{4+8+4} - \frac{e^{-2x}}{4+8+4}$$

* $\sin ix = i\sinh x$
$\cos ix = \cosh x$

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$$\frac{e^{2x}}{4+8+4} - \frac{e^{-2x}}{4-8+4} = 0$$

$$\frac{e^{2x}}{4+8+4} - \frac{x\,e^{-2x}}{8D+4} = 0$$

$$P.I = \frac{e^{2x}}{4+8+4} - \frac{x^2 e^{-2x}}{2}$$

DATE : / 20
PAGE No

for solution.

Add P.I and CF

Solution is

$$\left(c_1 e^{-2x} + c_2 x e^{-2x}\right) + \left(\frac{e^{2x}}{16} - \frac{x^2 e^{-2x}}{2}\right)$$

Ans

Rule: 2

if $R(x) = \sin(ax+b)$ or $\cos(ax+b)$

$$P.I = \frac{R(x)}{f(D)}$$

$$P.I = \frac{\sin(ax+b)}{f(D)}$$

$$= \frac{\sin(ax+b)}{f(D^2)}$$

$$P.I = \frac{\sin(ax+b)}{f(-a^2)} \quad f(-a)^2 \neq 0$$

if $f(-a^2) = 0$

then $P.I = x\frac{\sin(ax+b)}{f'(D^2)}$.

DATE: / 20
PAGE No

Ex: $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = e^{2x} + \cos x$

Let $\frac{d}{dx} = D$

$D^2y - 4Dy + 3y = 0$

$(D^2 - 4D + 3)y = 0$

$m^2 - 4m + 3 = 0$

$m^2 - m - 3m + 3 = 0$

$m(m-1) - 3(m-1) = 0$

$(m-1)(m-3) = 0$

$m = 1, 3$

y is non-zero Quantity not shows dependency

$C.f = C_1 e^{1x} + C_2 e^{3x}$

$P.I = \frac{e^{2x} + \cos x}{D^2 - 4D + 3}$

$= \frac{e^{2x}}{D^2 - 4D + 3} + \frac{\cos x}{D^2 - 4D + 3}$

$= \frac{e^{2x}}{4 - 8 + 3} + \frac{\cos x}{-1^2 - 4D + 3}$

$= \frac{e^{2x}}{-1} + \frac{\cos x}{-4D + 2}$

$= -e^{2x} + \frac{\cos x}{-2(2D - 1)}$

DATE : / 20
PAGE No

$$= -e^{2x} + \frac{\cos x\,(2D+1)}{2(2D-1)(2D+1)}$$

$$-e^{2x} - \frac{\cos x\,(2D+1)}{2(4D^2-1)}$$

$$= -e^{2x} - \frac{1}{2}\left(\frac{-2\sin x + \cos x}{(-4-1)}\right)$$

$$= -e^{2x} + \frac{1}{2}\left(\frac{-2\sin x + \cos x}{(-5)}\right)$$

$$= -e^{2x} + \frac{1}{10}(\cos x - 2\sin x)$$

Complete solution

$$y = CF + PI$$

$$y = (c_1 e^{x} + c_2 e^{3x}) + (-e^{2x}) + \frac{1}{10}(\cos x - 2\sin x)$$

$(1+x)^{-1} = 1 - x + x^2 - x^3$

$(1-x)^{-1} = 1 + x + x^2 + x^3$

DATE: / 20
PAGE No

$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}$

Rule III :- If $R(x) = x^n$

then,

$$P.I = \frac{x^n}{f(D)}$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$$

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \dots$$

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \dots$$

$$P.I = x^n [f(D)]^{-1}$$

Ex! $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 3y = \cos x + x^2$

$D^2y - 2Dy + 3y = 0$

$(D^2 - 2D + 3)y = 0$

$m^2 - 2m + 3 = 0$

$m = \frac{2 \pm \sqrt{4 - 4 \times 1 \times 3}}{2}$

$m = \frac{2 \pm 2\sqrt{2}}{2}$

$m = 1 \pm \sqrt{2}i$

Self $C.F = e^x(C_1 \cos\sqrt{2}x + C_2 \sin\sqrt{2}x)$ —

≥ 2 order ≥ linear factor Concept

DATE : / 20
PAGE No

$$P.I = \frac{\cos x + x^2}{f(D)} = \frac{\cos x + x^2}{D^2 - 2D + 3}$$

$$= \frac{\cos x}{D^2 - 2D + 3} + \frac{x^2}{D^2 - 2D + 3}$$

$$= \frac{\cos x}{-1^2 - 2D + 3} + \frac{x^2}{3\left(1 + \left(\frac{D^2 - 2D}{3}\right)\right)}$$

$$= \frac{\cos x}{-2D + 2} + \frac{1}{3}\left(1 + \left(\frac{D^2 - 2D}{3}\right)\right)^{-1} x^2$$

$$= \frac{\cos x(D+1)}{-2(D+1)(D+1)} + \frac{1}{3}\left\{1 - \left(\frac{D^2 - 2D}{3}\right) + \left(\frac{D^2 - 2D}{3}\right)^2\right\} x^2$$

$$= -\frac{1}{2}\left(\frac{-\sin x + \cos x}{D^2 - 1}\right) + \frac{1}{3}\left\{\right.$$

$$= \frac{1}{4}(-\sin x + \cos x) + \frac{1}{3}\left\{1 - \frac{D^2}{3} + \frac{2D}{3} + \frac{4D^2}{9}\right\} x^2$$

$$= \frac{1}{4}(\cos x - \sin x) + \frac{1}{3}\left\{x^2 - \frac{D^2}{3}(x^2) + \frac{2D}{3}(x^2) + \frac{4}{9}D^2(x^2)\right\}$$

$$= \frac{1}{4}(\cos x - \sin x) + \frac{1}{3}\left\{x^2 - \frac{2}{3} + \frac{2}{3} \times 2x + \frac{4}{9}(2)\right\}$$

$$= \frac{1}{4}(\cos x - \sin x) + \frac{1}{3}\left(x^2 - \frac{2}{3} + \frac{4x}{3} + \frac{8}{9}\right)$$

DATE: / 20
PAGE No

Rule:(iv) If $R(x) = e^{ax} \cdot V(x)$ where $V(x)$ is any function of x

$$P.I = \frac{e^{ax} \cdot V(x)}{f(D)}$$

$D \to D+a$

$$= e^{ax} \left\{ \frac{V(x)}{f(D+a)} \right\}$$

Rule:(v) **(a)** If $R(x) = x^n \cdot \sin(ax+b)$ or $= x^n \cos(ax+b)$

$$P.I = \frac{x^n \sin(ax+b)}{f(D)}$$

$$= I.m \text{ of } \frac{x^n e^{i(ax+b)}}{f(D)}$$

$$= I.m \text{ of } e^{i(ax+b)} \left\{ \frac{x^n}{f(D+ai)} \right\}$$

(b) $R(x) = x^n \cdot V(x)$

$$P.I = \frac{x^n \cdot V(x)}{f(D)} =$$

DATE: / 20
PAGE No

$$P.I = \left\{ x^n - \frac{f'(D)}{f(D)} \right\} \frac{V(x)}{F(D)}$$

Rule (VI) General Method. Eg $R(x) = \phi(x)$ where $\phi(x)$ is any f^n of x.

$$(D^2 + PD + Q)\, y = \phi(x)$$

$$(D - m_1)(D - m_2)\, y = \phi(x)$$

$$P.I = \frac{\phi(x)}{(D - m_1)(D - m_2)}$$

$$P.I = \frac{1}{(D - m_1)} \left\{ \frac{\phi(x)}{(D - m_2)} \right\}$$

$$y = \frac{\phi(x)}{D - m_2}$$

$$(D - m_2)\, y = \phi(x)$$

$$Dy - m_2 y = \phi x$$

$$\frac{dy}{dx} - m_2 y = \phi x$$

$$I.F = e^{-\int m_2\, dx} = e^{-m_2 x}$$

$$y \times e^{-m_2 x} = \int Q(x)\, e^{-m_2 x}\, dx$$

$$y = e^{m_2 x} \int \phi(x) e^{-m_2 x}\, dx.$$

DATE : / 20
PAGE No

$$\frac{\phi(x)}{D-m_2} = e^{m_2 x}\int Q(x)\, e^{-m_2 x}\, dx$$

or

$$P.I = \frac{Q(x)}{D-a} = e^{ax}\int Q(x)\, e^{-ax}\, dx$$

Q. $(D^4-1)y = e^x \cos x$

$$P.I = \frac{e^x \cos x}{(D^4-1)}$$

$$= \frac{e^x \cos x}{(D^2-1)(D^2+1)}$$

$(D^2-1)(D^2+1)$

$D^2=1 \quad D^2=-1$

$D=\pm 1 \quad D^2=\pm i$

$$= e^x \cos x$$

A.E is $m^4-1=0$

$m = \pm 1, \; m = \pm i$

$$C.F = C_1 e^x + C_2 e^{-x} + C_3 \cos x + C_4 \sin x$$

$$P.I = \frac{e^x \cos x}{(D^4-1)}$$

$$= e^x \left\{ \frac{\cos x}{(D+1)^4-1} \right\}$$

DATE: / 20
PAGE No

$$= e^{x} \left\{ \frac{\cos x}{D^4 + 4D^3 + 6D^2 + 4D + (-1)} \right\}$$

$$= e^{x} \left\{ \frac{\cos x}{D^2 \cdot D^2 + 4D \cdot D^2 + 6D^2 + 4D} \right\}$$

$$= e^{x} \left\{ \frac{\cos x}{1 + 4D(-1) + 6(-1) + 4D} \right\}$$

$$= e^{x} \left\{ \frac{\cos x}{1 - 6 - 4D + 4D} \right\}$$

$$= \frac{e^{x}}{5} [\cos x]$$

Q. (2) $(D^2y + 2Dy + y) = \frac{e^{-x}}{(x+2)}$

$$P.I = \frac{e^{-x}}{(x+2)} \cdot \frac{1}{(D^2 + 2D + 1)}$$

$$= e^{-x} \left\{ \frac{1}{(D+1+1)^2} \right\} \frac{1}{(x+2)}$$

$$= e^{-x} \left\{ \frac{1}{D^2} \right\} \frac{1}{(x+2)}$$

DATE : / 20
PAGE No

$$e^{-x}\iint \frac{1}{(x+2)}\, dx\, dx.$$

Q. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x e^x \cos x$

$$(D^2 - 2D + 1)y = x e^x \cos x$$

$$P.I = \frac{x e^x \cos x}{D^2 - 2D + 1}$$

$$= \frac{x e^x \cos x}{(D-1)^2}$$

$$= e^x \left\{ \frac{x \cos x}{(D-1+1)^2} \right\}$$

$$= e^x \left\{ \frac{x \cos x}{(D)^2} \right\}$$

$$= e^x \frac{1}{D^2}(x \cos x)$$

$$= e^x \iint x \cos x\, dx\, dx.$$

Doubt

DATE: / 20
PAGE No

Q $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x \sin x$

$P.I = \frac{x \sin x}{(D^2 - 2D + 1)}$

$P.I = I.P \text{ of } \frac{x e^{ix}}{(D-1)^2}$

$= I.P \text{ of } e^{ix} \left\{ \frac{x}{(D+i-1)^2} \right\}$

$= I.P \text{ of } e^{ix} \left\{ \frac{x}{D^2 + (-1) + 1 + 2iD - 2D - 2i} \right\}$

$= I.P \text{ of } e^{ix} \left\{ \frac{x}{D^2 + 2D(i-1) - 2i} \right\}$

take common $-2i$

$= I.P \text{ of } \frac{e^{ix}}{-2i} \left\{ 1 - \left(\frac{D^2 + 2D(i-1)}{2i} \right) \right\}^{-1} x$

$= I.P \text{ of } \frac{e^{ix}}{-2i} \left\{ 1 + \frac{1}{2i}(D^2 + 2D(i-1)) \right\} x.$

$= I.P \text{ of } \frac{e^{ix}}{-2i} \left\{ x + \frac{2}{2i}(0 + 2(i-1)) \right\}.$

$= I.P \text{ of } \frac{e^{ix}}{-2i} \{ x$

DATE : / 20
PAGE No

$$= \text{I.P of } \frac{e^{ix}}{-2i}(x+1+i)$$

$$= \text{I.P of } \frac{ie^{ix}}{2}\{(x+1)+i\}$$

$$= \text{I.P of } \frac{1}{2}(\cos x + i\sin x)\{i(x+1)-1\}$$

$$= \frac{1}{2}\{\cos x\, i(x+1) - \sin x\}$$

Ques $\frac{d^2y}{dx^2} + a^2y = \sec ax.$

Sol^n: $(D^2+a^2)y = \sec ax$

A.E is $(D=m)$

$$m^2+a^2=0$$
$$m = \pm ia$$

$$C.F = C_1\cos ax + C_2 \sin ax$$

$$P.I = \frac{\sec ax}{D^2+a^2}$$

$$= \frac{\sec ax}{(D+ia)(D-ia)}$$

$$= \frac{1}{2ai}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right] \sec ax$$

$$P.I = \frac{1}{2ai}\left[\frac{\sec ax}{D-ia} - \frac{\sec ax}{D+ia}\right]$$

$$I_1 \qquad I_2$$

$$I_1 = \frac{\sec ax}{D-ia} = e^{iax}\int e^{-iax}\sec ax\, dx$$

$$I_1 = e^{iax}\int (\cos ax + i\sin(-ax))\sec ax\, dx$$

$-i\sin ax$

$$I_1 = e^{iax}\int (1 + (-i\tan ax))\, dx$$

$$I_1 = e^{iax}\left[x - \frac{i}{a}\log \sec x\right]$$

similarly

$$I_2 = e^{-iax}\left[x + \frac{i}{a}\log \sec x\right]$$

$$P.I = \frac{1}{2ai}\left[x\left(e^{iax} - e^{-iax}\right) - \frac{1}{a}\log \sec ax\left(e^{iax} + e^{-iax}\right)\right]$$

we know $\cos ax = \dfrac{e^{iax} + e^{-iax}}{2}$

$$\sin ax = \frac{e^{iax} - e^{-iax}}{2i}$$

DATE : / 20
PAGE No

$$P.I = \frac{1}{2ai}\left[x\,2i\sin ax - \frac{i}{a}\log\sec ax\,(2\cos ax)\right]$$

$$P.I = \frac{1}{a}\left[x\sin ax - \frac{1}{a}(\log\sec ax)(\cos ax)\right]$$

Solution of the eq^n :-

$$y = c_1\cos ax + c_2\sin ax + \frac{1}{a}\left[x\sin ax - \frac{1}{a}(\log\sec ax)(\cos ax)\right]$$

* Cauchy's homogenious linear d.E

A d.E of the form

$$a_0x^n\frac{d^ny}{dx^n} + a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \cdots$$

$$\cdots \text{any} = R(x)$$

where $a_0, a_1, a_2 \cdots a_n$ are constant.

$$x = e^z \Rightarrow z = \log_e x$$

and $D = \frac{d}{dz}$

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}$$

$$\frac{dy}{dx} = \frac{dy}{dz}\times\frac{1}{x}$$

$$x\frac{dy}{dx} = Dy$$

Also $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$= \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$

$= \frac{d}{dx}\left(e^{-z}\frac{dy}{dz}\right)$

$= \frac{d}{dz}\left(e^{-z}\frac{dy}{dz}\right)\frac{dz}{dx}$

$$\frac{d^2y}{dx^2} = \left\{e^{-z}\frac{d^2y}{dz^2} + \frac{dy}{dz}(-e^{-z})\right\}\frac{1}{x}$$

$= e^{-z}\left\{D^2y - Dy\right\}\frac{1}{x}$

$= \frac{1}{x^2}\left[D^2y - Dy\right]$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

DATE : / 20
PAGE No

Que Solve the D.E

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$$

(1) Let $x = e^z \Rightarrow z = \log x$, $D = \frac{d}{dz}$

$$D(D-1)y - 2Dy - 4y = x^4 e^{4z}$$

then eq' can be written as

$$(D^2 - D - 2D - 4)y = e^{4z}$$

$$(D^2 - 3D - 4)y = e^{4z}$$

A.E is $m^2 - 3m - 4 = 0$

$$m = 4, -1$$

$$CF = C_1 e^{4z} + C_2 e^{-z}$$

$$P.I = \frac{e^{4z}}{(D^2 - 3D - 4)}$$

$$= \frac{e^{4z}}{16 - 12 - 4} = \frac{z e^{4z}}{2D - 3}$$

$$= \frac{z e^{4z}}{8 - 3} = \frac{z e^{4z}}{5}$$

Sol^n is $y = C_1 e^{4z} + C_2 e^{-z} + \frac{z e^{4z}}{5}$

DATE: / 20
PAGE No

Put $x = e^z \Rightarrow z = \log_e x$

$$y = c_1 x^4 + \frac{c_2}{x} + (\log x)\frac{x^4}{5}$$

* Legendre's Homogeneous linear d.e:

A d.e of the form

$$a_0(ax+b)^n \frac{d^n y}{dx^n} + a_1(ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2(ax+b)^{n-2} \frac{d^{n-2} y}{dx^{n-2}}$$

$$+ - - - - - - + a_{n-1}(ax+b)\frac{dy}{dx} + a_n y$$

$$= R(x).$$

where $a_0, a, b, a_1, a_2, \ldots a_n$ are constants

If $a=1$, and $b=0$ then the equation is reduced to Cauchy's Homogeneous d.e

Let $ax+b = e^z$.

$z = \log(ax+b)$ and $D = \frac{d}{dz}$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$\frac{dy}{dx} = \frac{dy}{dz}\left(\frac{a}{ax+b}\right).$$

DATE : / 20
PAGE No

$$(ax+b)\frac{dy}{dx} = aDy$$

$$(ax+b)\frac{dy}{dx} = a^2D(D-1)y$$

Q. Solve d.e

$$(2x+1)^2\frac{d^2y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12y = 16x$$

Let $2x+1 = e^z$

$$z = \log_e(2x+1)$$

$$D = \frac{d}{dz}$$

$$2^2D(D-1)y - 2(2)Dy - 12y = 16\left(\frac{e^z-1}{2}\right)$$

$$4D(D-1)y - 4Dy - 12y = 16\left(\frac{e^z-1}{2}\right)$$

$$(4D^2-4D)y - 4Dy - 12y = 16\left(\frac{e^z-1}{2}\right)$$

$$(4D^2-4D-4D-12)y = 16\left(\frac{e^z-1}{2}\right)$$

$$(4D^2-8D-12)y = 16\left(\frac{e^z-1}{2}\right)^2$$

$$(D^2-2D-3)y = 4\left(\frac{e^z-1}{2}\right)$$

DATE: / 20
PAGE No

$$(D^2-2D-3)y = 2(e^{2z}+1)$$

$$m^2-2m-3=0$$
$$m^2+m-3m-3=0$$
$$m(m+1)-3(m+1)=0$$
$$m=3,-1$$

$$C.F = C_1e^{3z} + C_2e^{-1z}$$

$$P.I = \underline{2(e^{2z}+1)}$$

Que: $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log x \sin(\log x)$

$x = e^z \qquad D = \frac{d}{dz}$

$z = \log x^2 =$

$$(D(D-1)y + Dy + y) = \log x \sin(\log x)$$

$$(D(D-1) + D + 1]y = \log x \sin(\log x)$$
$$= z\sin z$$

$$(D^2-D+D+1)y = \log x \sin(\log x)$$
$$= z\sin z$$

$$(D^2+1)y = \log x \sin(\log x)$$
$$= z\sin z$$

A.E: (m^2+1)

$$C.f = m^2+1=0$$
$$m^2 = -1$$
$$m = \pm i$$

DATE : / 20
PAGE No

$$C.F = C_1 \cos z + C_2 \sin z)$$

$$P.I = \frac{z \sin z}{D^2+1}$$

$$P.I = I.P \text{ of } \frac{z \cdot e^{iz}}{D^2+1}$$

$$= I.P \text{ of } e^{iz} \left\{ \frac{z}{(D+i)^2+1} \right\}$$

$$= I.P \text{ of } e^{iz} \left\{ \frac{z}{D^2+i^2+2iD+1} \right\}$$

$$= I.P \text{ of } e^{iz} \left\{ \frac{z}{D^2+2iD+1} \right\}$$

$$= I.P \text{ of } e^{iz} \left\{ \frac{z}{D^2+2iD} \right\}$$

$$= I.P \text{ of } \frac{e^{iz}}{2iD} \left\{ \frac{z}{1+\frac{D^2}{2iD}} \right\}$$

$$= I.P \text{ of } \frac{e^{iz}}{2iD} \left[z \left(1+\frac{D}{2i}\right)^{-1} \right].$$

DATE: / 20
PAGE No

$$= \text{I.P of } \frac{e^{iz}}{2iD}\left[1 - \frac{D}{2i} - \frac{D^2}{4}\right] z$$

$$= \text{I.P of } \frac{e^{iz}}{2iD}\left[\frac{1}{D}(z) - \frac{1}{2i} \times z - \frac{D}{4}(z)\right]$$

$$= \text{I.P of } \frac{e^{iz}}{2i}\left[\frac{z^2}{2} - \frac{z}{2i} - \frac{2}{4}\right]$$

$$= \text{I.P of } \frac{e^{iz}}{2i}\left[\frac{z^2}{2} + \frac{z}{2}i - \frac{1}{4}\right]$$

$$= -\frac{1}{2}(\cos z + i\sin z)\left\{\left(\frac{z^2}{2} - \frac{1}{4}\right) + \frac{z}{2}\right\}$$

$$\text{P.I} = -\frac{1}{2}\left\{\left(\frac{z^2}{2} - \frac{1}{4}\right)\cos z - \frac{z}{2}\sin z\right\}.$$

$$y = \text{C.F} + \text{P.I}$$

$$y = C_1\cos z + C_2 \sin z - \frac{1}{2}\left\{\left(\frac{z^2}{2} - \frac{1}{4}\right)\cos z - \frac{z}{2}\sin z\right\}$$

DATE : / 20
PAGE No

Sol^n of eq^n (1)

$$y = C_1\cos(\log x) + C_2\sin(\log x) - \frac{1}{2}\left\{\left(\frac{(\log x)^2}{2} - \frac{1}{4}\right)\cos(\log x) - \frac{1}{2}\log x \cdot \sin(\log x)\right\}$$

Que: Solve the d.e

$$(2x+1)^2\frac{d^2y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12y = 16x \quad (1)$$

$2x+1 = e^z$

$z = \log(2x+1)$ $\qquad D = \frac{d}{dz}$

$$2D(D-1)y - 2Dy - 12y = 16x$$

$$(2D(D-1) - 2D - 12)y = 16x$$

$$(2D^2 - 4D - 4D - 12)y = 16x$$

$$(4D^2 - 8D - 12)y = 16x$$

$$(D^2 - 2D - 3)y = \frac{16}{4}\left(\frac{e^z - 1}{2}\right)$$

2 | 64
2 | 32
2 | 16
2 | 8
2 | 4
2 | 2
1

DATE: / 20
PAGE No

$(D^2-2D-3)y = 2(e^x+1)$ — (1)

A.E $= m^2-2m-3=0$

$m = -1, 3$

C.F $= c_1e^{-x} + c_2e^{3x}$

P.I $= \dfrac{2(e^x+1)}{(D^2-2D-3)}$

$= \dfrac{2e^x}{D^2-2D-3} - \dfrac{2e^{0x}}{D^2-2D-3}$

$= \dfrac{2e^x}{1-2-3} - \dfrac{2e^{0x}}{0-0-3}$

$= \dfrac{2e^x}{-4} - \dfrac{2}{3}$

$= -\dfrac{e^x}{2} + \dfrac{2}{3}$

$y = c_1e^{-x} + c_2e^{3x} + \left(-\dfrac{e^x}{2} + \dfrac{2}{3}\right)$ Ans

$y = c_1e^{-(\log(2x+1))} + c_2e^{3(\log(2x+1))} + \left(-\dfrac{(2x+1)}{2} + \dfrac{2}{3}\right)$

$y = \dfrac{c_1}{2x+1} + \dfrac{c_2}{(2x+1)^3} - \dfrac{1}{2}(2x+1) + \dfrac{2}{3}$

DATE : / 20
PAGE No

* Solution of Second Order Linear D.E :

1) Solution of Second Order Linear D.E with variable coefficient :

Consider a differential eq^n of the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \text{--- (i)}$$

where P and Q are the fn of x eq^n (i) is a second order linear D.E of variable coefficient in standard form

M-1 If one Part of solution is known or C.F solution is given (changing the dependent variable)

(493)

Given $y = u$ be the C.F solution of eq^n (i)

$$\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0 \quad \text{--- (ii)}$$

Let v be the Remaining sol^n of eq^n (i)

V coefficient always zero
if one sol^n is known

DATE: / / 20
PAGE No

then Complete solution of eq (i)

$$y = u.v. \quad \text{(iii)}$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + \frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2} + \frac{dv}{dx}\frac{du}{dx}$$

$$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$$

Put the value of $y, \frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in eq (i)

$$u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2} + P\left(u\frac{dv}{dx} + v\frac{du}{dx}\right) + Q(uv) = R.$$

$$u\frac{d^2v}{dx^2} + \left(2\frac{du}{dx} + Pu\right)\frac{dv}{dx} + \left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right)v = R$$

$$u\frac{d^2v}{dx^2} + \left(2\frac{du}{dx} + Pu\right)\frac{dv}{dx} = R$$

$$\frac{d^2v}{dx^2} + \left(\frac{2}{u}\frac{du}{dx} + P\right)\frac{dv}{dx} = \frac{R}{u}$$

Let $\frac{dv}{dx} = z$ then $\frac{d^2v}{dx^2} = \frac{dz}{dx}$

DATE : / 20
PAGE No

$$\frac{dz}{dx} + \left(\frac{2}{y}\frac{dy}{dx} + P\right) z = \frac{R}{y} \quad \text{--- (iii)}$$

Find value of z and put in eq^n (iii) and integrate

Eg: Solve

$$x^2\frac{d^2y}{dx^2} - (x^2 + 2x)\frac{dy}{dx} + (x+2)y = x^3e^x.$$

Given that $y = x$ is a solution.

Soln:

$$\frac{d^2y}{dx^2} - \left(\frac{x^2+2x}{x^2}\right)\frac{dy}{dx} + \left(\frac{x+2}{x^2}\right)y = \frac{xe^x}{\phi}$$

Given $y = x$ be the c.f soln and let v be the Remaining soln of eq^n (i)

To complete soln of eq^n (i) is

$$y = x \cdot v \quad \text{--- (ii)}$$

$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

$$\frac{d^2y}{dx} = x\frac{d^2v}{dx} + \frac{dv}{dx} + \frac{dv}{dx}$$

$$\frac{d^2y}{dx} = x\frac{d^2v}{dx} + 2\frac{dv}{dx}$$

DATE: / 20
PAGE No

from (i)

$$x\frac{d^2v}{dx} + 2\frac{dv}{dx} - \frac{(x^2+2x)}{x^2}\left(x\frac{dv}{dx}+v\right) +$$

$$+\left(\frac{x+2}{x^2}\right)uv = xe^x$$

$$x\frac{d^2v}{dx^2} + \left(\frac{2-(x^2+2x)}{x}\right)\frac{dv}{dx} + \left(\frac{-(x+2)}{x}\right)v$$

$$x\frac{d^2v}{dx^2} + \left(2-x-2\right)\frac{dv}{dx} + v\left(-1-\frac{2}{x}+\frac{x+2}{x^2}\right) = xe^x$$

$$x\frac{d^2v}{dx^2} + \left(-x\frac{dv}{dx}\right) + \left(\frac{-x-2}{x}+\frac{x+2}{x}\right) = xe^x$$

$$\frac{d^2v}{dx^2} + \left(-\frac{dv}{dx}\right) = e^x \quad -(iii)$$

let $\frac{dv}{dx} = z \Rightarrow \frac{d^2v}{dx^2} = \frac{dz}{dx}$

$$\frac{dz}{dx} - z = e^x$$

$$I.F = e^{\int -dx} = e^{-x}$$

sol^n $\quad z\,e^{-x} = \int e^x \cdot e^{-x}\,dx + c$

$$z\,e^{-x} = x + c_1 \Rightarrow z = xe^x + c_1 e^x$$

DATE : / 20
PAGE No

$$\frac{dv}{dx} = xe^x + c_1 e^x$$

$$v = \int x e^x dx + c_1 \int e^x dx + c_2$$

$$v = e^x(x-1) + c_1 e^x + c_2$$

Complete sol^n

$$\boxed{y = (x^2 - x)e^x + c_1 x e^x + c_2 x}$$

* To obtain the C.F solution of a diff. eq^n

(i) $1 + P + Q = 0$ then
$u = e^x$.

(ii) if $1 - P + Q = 0$, then $u = e^{-x}$

(iii) if $1 + \frac{P}{a} + \frac{Q}{a^2} = 0$ then
$u = e^{ax}$

(iv) if $P + Qx = 0$ then $u = x$

(v) if $P^2 + 2Px + Qx^2 = 0$
then $u = x^2$

DATE: / 20
PAGE No

(vi) If $n(n-1) + npx + Qx^2 = 0$

then, $y = x^n$

Qu $\sin^2 x \frac{d^2y}{dx^2} = 2y$

given that

$y = \cot x$ is a sol^n.

Sol^n $\sin^2 x \frac{d^2y}{dx^2} - 2y = 0$

on comparing standard eq^n

$P = 0, \quad Q = \frac{-2}{\sin^2 x}, \quad R = 0$

$y = \cot x$

let v the remaining solution of the given eq^n (1)

$y = uv = \cot x \cdot v$

$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$

$\frac{d^2v}{dx^2} + \left(0 + \frac{2}{\cot x}(-\operatorname{cosec}^2 x)\right)\frac{dv}{dx} = 0$

DATE : / 20
PAGE No

$$\frac{d^2v}{dx^2} - 2\frac{\sin x}{\cos x}\operatorname{cosec}^2 x \frac{dv}{dx} = 0$$

Let $\frac{dv}{dx} = z$ $\quad \frac{d^2v}{dx^2} = \frac{dz}{dx}$

$$\frac{dz}{dx} - \left(2\frac{\sin x}{\cos x}\operatorname{cosec}^2 x\right) z = 0$$

$$\frac{dz}{dx} + \left(\frac{2}{\sin x \cos x}\right) z = 0$$

$$\frac{dz}{dx} - \frac{4}{\sin 2x} dx = 0$$

$$\log z - 4\int \operatorname{cosec} 2x \, dx = 0$$

$$\log z - \frac{4}{2}\log(\operatorname{cosec} 2x - \cot 2x) + \log c_1$$

$$\log z = \log(\operatorname{cosec} 2x - \cot 2x)^2 c_1$$

$$z = c_1(\operatorname{cosec} 2x - \cot 2x)^2$$

$$\frac{dv}{dx} = c_1\left(\frac{1}{\sin 2x} - \frac{\cos 2x}{\sin 2x}\right)^2$$

$$\frac{dv}{dx} = c_1\left(\frac{1-\cos 2x}{\sin 2x}\right)^2$$

$$\frac{dv}{dx} = c_1\left(\frac{2\sin^2 x}{2\sin x \cos x}\right)^2$$

DATE: / 20
PAGE No

$$dv = c_1(\tan^2 x)\,dx$$

$$\int dv = c_1 \int \tan^2 x\,dx + c$$

$$\int dv = c_1 \int (\sec^2 x - 1)\,dx$$

$$v = c_1(\tan x - x) + c_2$$

C.S is

$$y = \cot x\,[c_1(\tan x - x) + c_2]$$

$$y = c_1(1 - x\cot x) + c_2 \cot x$$

Q.2 $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$

Given $y = x + \frac{1}{x}$

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = 0$$

$y = x + \frac{1}{x}$ Let v is another soln

$y = u \cdot v = \left(x + \frac{1}{x}\right) v$

$P = \frac{1}{x}$, $Q = \frac{-1}{x^2}$, $R = 0$

Also $y = x + \frac{1}{x}$

DATE : / 20
PAGE No

Let v be the remaining sol^{n} of eq^{n} (1)

C.S is

$$y = (x + \tfrac{1}{x})v \quad -(i)$$

eq^{n}(1) Can be written as

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

$$\frac{d^2v}{dx^2} + \left(\frac{1}{x} + \frac{2}{(x+\frac{1}{x})}\left(1 - \frac{1}{x^2}\right)\right)\frac{dv}{dx} = 0$$

$$\frac{d^2v}{dx^2} + \left(\frac{1}{x} + \frac{2}{x+1}\left(\frac{x^2-1}{x^2}\right)\right)\frac{dv}{dx} = 0$$

Let $\frac{dv}{dx} = z$

$$\frac{d^2v}{dx^2} = \frac{dz}{dx}$$

$$\frac{dz}{dx} + \left(\frac{1}{x} + \left(\frac{2}{x+1}\right)\left(\frac{x^2-1}{x}\right)\right)z = 0$$

$$\frac{dz}{dx} + \left(\frac{1}{x} + \frac{2(x^2-1)}{x(x+1)}\right)z = 0$$

$$\int \frac{dz}{z} + \int \left(\frac{1}{x} + \frac{2(x^2-1)}{x(x+1)}\right)dx = 0$$

DATE: / / 20
PAGE No

$$\int \frac{dz}{z} + \int \left(\frac{1}{x} + \frac{2(x^2+1) - 4}{x(x^2+1)} \right) dx = 0$$

$$\log z + \log x + \int \left(\frac{2}{x} - \frac{4}{x(x^2+1)} \right) dx = 0$$

$$\log z + \log x + 2\log x - \int \frac{4x}{x^2(x^2+1)} dx + \log c_1$$

$$\log z + \log x + 2\log x - \int 4x \left(\frac{1}{x^2} - \frac{1}{x^2+1} \right) dx + \log c_1$$

$$\log z + 3\log x - 4 \int \left(\frac{1}{x} - \frac{x}{1+x^2} \right) dx - \log c_1 = 0$$

$$\log z + 3\log x - 4\log x + 2 \int \frac{2x}{1+x^2} dx - \log c_1 = 0$$

$$\log z + 3\log x - 4\log x + 2\log(1+x^2) - \log c_1 = 0$$

$$\log z - \log x + 2\log(1+x^2) = \log c_1$$

$$\log z = \log x - \log(1+x^2) + \log c_1$$

$$\log z = \log \frac{ax}{(1+x^2)^2}$$

$$z = \frac{c_1 x}{(1+x^2)^2}$$

DATE : / 20
PAGE No

$$\frac{dv}{dz} = \frac{cx}{(1+x^2)^2}$$

$$\int dv = \int \frac{cx}{(1+x^2)^2} dx$$

$$v = \int \frac{c_1 x}{(1+x^2)^2} dx$$

Method-2

Changing the dependent variable

(~~Normal form~~)

(Normal form) or Removal of 1st Derivative

Consider a second order linear D.E

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \text{--- (1)}$$

Let $y = u \cdot v$ be the complete sol^n of eq^n (1)

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

DATE: / 20
PAGE No

$$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + \frac{dv}{dx}\frac{du}{dx} + v\frac{d^2u}{dx^2} + \frac{dv}{dx}\frac{du}{dx}$$

$$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\cdot\frac{dv}{dx} + v\frac{d^2u}{dx^2}$$

Put y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in eq^n (1)

$$u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\cdot\frac{dv}{dx} + v\frac{d^2u}{dx^2}\Big) + P\Big($$

$$P\left(u\frac{dv}{dx} + v\frac{du}{dx}\right) + Q(u\cdot v) = R$$

$$u\frac{d^2v}{dx^2} + \left(2\frac{du}{dx} + P\cdot u\right)\frac{dv}{dx} + \left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right)v = R$$

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} + v\left(\frac{1}{u}\frac{d^2u}{dx^2} + \frac{P}{u}\frac{du}{dx} + Q\right) = \frac{R}{u}$$

⇒ To Remove 1st derivative term in eq^n (ii)

Put

$$P + \frac{2}{u}\frac{du}{dx} = 0$$

$$\frac{2}{u}\frac{du}{dx} = -P$$

$$\int\frac{du}{u} = \int\frac{-P}{2}\,dx$$

DATE : / 20
PAGE No

$$\log u = -\int \frac{P}{2}\,dx$$

$$u = e^{-\int \frac{P}{2}\,dx}$$

Now eq^{n} (ii) Become

$$\frac{d^2v}{dx^2} + v\left(\frac{1}{u}\frac{d^2u}{dx^2} + \frac{P}{u}\frac{du}{dx} + Q\right) = \frac{R}{u} \quad \text{---(iii)}$$

Normal eq^{n}

$$\frac{du}{dx} = -\frac{P}{2}u$$

$$\frac{d^2u}{dx^2} = -\frac{1}{2}\left(P\frac{du}{dx} + u\frac{dP}{dx}\right)$$

$$\frac{d^2u}{dx^2} = -\frac{1}{2}\left(P\left(\frac{-Pu}{2}\right) + u\left(\frac{dP}{dx}\right)\right)$$

$$= \frac{1}{4}P^2u - \frac{1}{2}u\frac{dP}{dx}$$

eq^{n}

$$\frac{d^2v}{dx^2} + v\left(\frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} + \frac{P}{4}\left(\frac{-Pu}{2}\right) + Q\right) = \frac{R}{u}$$

DATE: / 20
PAGE No

$$\frac{d^2v}{dx^2} + v\left(\frac{P^2}{4} - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{2} + Q\right) = \frac{R}{4}$$

$$\frac{d^2v}{dx^2} + v\left(-\frac{P^2}{4} - \frac{1}{2}\frac{dP}{dx} + Q\right) = \frac{R}{4}$$

$$\boxed{\frac{d^2v}{dx^2} + Q_1 v = R_1}$$ — Normal form e_1^4.

$$Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$$

$$R_1 = \frac{R}{4}$$

Q_1 = Constant or $\frac{\text{Const}}{x^2}$ Always

Q. ~~$y\,dx - x\,dy - (\log x)\,dx = 0$~~

~~$(y - \log x)\,dx - x\,dy = 0$~~

DATE : / 20
PAGE No

Q $\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} - 5y = 0$ — (1)

$P = -2\tan x, \quad Q = -5, \quad R = 0$

$u = e^{-\int \frac{P}{2} dx}$

$\frac{d^2v}{dx^2} + Q_1 v = R_1$ — (ii)

where $Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$ and

$R_1 = \frac{R}{u}$

$u = e^{+\int \frac{2\tan x}{2} dx} = u = e^{\int \tan x dx}$

$u = e^{\log(\sec x)}$.

$u = \sec x$

$Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$

$= -5 - \frac{1}{2}\left(\frac{d}{dx}(-2\tan x dx)\right) - \frac{4\tan^2 x}{4}$

$Q_1 = -5 + \sec^2 x - (\sec^2 x - 1)$

$Q_1 = -5 + 1 = -4$ Ans.

$Q_1 = -4$

DATE: / 20
PAGE No

$R_1 = \frac{R}{4} = 0$

From (ii)

$\frac{d^2v}{dx^2} + (-4)v = 0$

$(D^2 - 4)v = 0$

D.E $(m-2)(m+2)$

$m = -2, -2$

$C.F = c_1 e^{2x} + c_2 e^{-2x}$

$v = C.F + P.I = c_1 e^{2x} + c_2 e^{-2x}$

Complete sol^n of eq^n (i) is

$y = u \cdot v$

$y = \sec x (c_1 e^{2x} + c_2 e^{-2x})$.

Q $\frac{d^2y}{dx^2} - \frac{1}{\sqrt{x}} \frac{dy}{dx} + \frac{y}{4x^2}(-8 + \sqrt{x} + x) = 0$

$P = \frac{-1}{\sqrt{x}}$, $Q = \left(\frac{-8 + \sqrt{x} + x}{4x^2}\right)$, $R = 0$

$u = e^{\int \frac{P}{2} dx}$ then the eq^n (1) can be written as

DATE : / 20
PAGE No

Normal form

$$\frac{d^2v}{dx^2} + Q_1 v = R_1 \quad \text{where}$$

$$Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4} \quad \text{and} \quad R_1 = \frac{R}{4}$$

$$u = e^{\int \frac{1}{2x} dx} \Rightarrow u = e^{\frac{1}{2}\log x}$$

$$Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$$

$$= -\frac{8 + \sqrt{x} + x}{4x^2} + \frac{1}{2}\left(\frac{d}{dx}(x^{-1/2})\right) - \frac{1}{4x}$$

$$Q_1 = -\frac{8}{4x^2} + \frac{\sqrt{x}}{4x^2} + \frac{x}{4x^2} + \frac{1}{2}\times\left(\frac{-1}{2}\frac{1}{x^{3/2}}\right) - \frac{1}{4x}$$

$$Q_1 = -\frac{2}{x^2} + \frac{1}{4x^{3/2}} + \frac{1}{4x} - \frac{1}{4}\frac{1}{x^{3/2}} - \frac{1}{4x}$$

$$Q_1 = \frac{-2}{x^2}$$

$$R_1 = \frac{R}{4} = 0$$

from eq^n (11)

DATE: / 20
PAGE No

$$\frac{d^2v}{dx^2} - \frac{2}{x^2}v = 0$$

$$x^2\frac{d^2v}{dx^2} - 2v = 0$$

Let $x = e^z \Rightarrow z = \log_e x$

$D = \frac{d}{dz}$ then

eq (iii) can be written as

$$D(D-1)v - 2v = 0$$
$$(D^2+1)v - 2v = 0$$
$$(D^2 - D - 2)v = 0$$

A.E is $m^2 - m - 2 = 0$

$m = +2, -1$

$C.F = C_1e^{2z} + C_2e^{-z}$

$P.I = 0$

$v = C_1e^{2x} + C_2e^{-z}$

solution of eq (ii)

$v = C_1x^2 + \frac{C_2}{x}$

$y = e^{\sqrt{x}}$

DATE : / 20
PAGE No

Complete sol^{n} of eq^{n} (1) is

$y = uv$

$= e^{\frac{x}{2}}\left(c_1 x^2 + \frac{c_2}{x}\right)$ Ans

Prob

Sol^{n} $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1)y = -3e^{x^2}\sin 2x$

$P = -4x$, $Q = 4x^2 - 1$

$R = -3e^{x^2}\sin 2x$

Let $u = e^{-\int \frac{P}{2} dx}$ then eq^{n} (1) can be written as

$\frac{d^2v}{dx^2} + v = R_1$

$u = e^{-\int \frac{P}{2} dx}$

$u = e^{x^2}$

$u = e^{x^2}$ ✓

DATE: / 20
PAGE No

$$Q_1 = Q - \frac{1}{Q}\frac{dP}{dx} - \frac{P^2}{Q}$$

$$= (4x^2 - 1) - \frac{1}{2}(-4) - \frac{16x^2}{4}$$

$$= 4x^2 - 1 + 2 - 4x^2$$

$$Q_1 = 1 \qquad R_1 = -3\sin 2x$$

$$\frac{d^2v}{dx^2} + v = -3\sin 2x$$

$$A.E = m^2 + 1 = 0$$

$$m = \pm i$$

$$C.F = C_1 \cos x + C_2 \sin x$$

$$P.I = \frac{-3\sin 2x}{D^2 + 1}$$

$$P.I = \frac{3\sin 2x}{-4 + 1} = \sin 2x$$

$$v_1 = C_1 \cos x + C_2 \sin x + \sin 2x$$

$$y = e^{x^2}(C_1 \cos x + C_2 \sin x + \sin 2x)$$

DATE : / 20
PAGE No

Method-3:

Changing the independent variable:

Consider

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

P, Q, and R are the fn of x

Here we change the independent variable x to independent variable z. by substitution

$$z = f(x)$$

Now $$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}$$

also

$$\frac{d^2y}{dx^2} = \frac{dy}{dx}\left(\frac{dy}{dz}\right)$$

$$= \frac{d}{dx}\left(\frac{dy}{dz}\cdot\frac{dz}{dx}\right)$$

$$= \frac{d}{dz}\left(\frac{dy}{dz}\cdot\frac{dz}{dx}\right)\frac{dz}{dx}$$

DATE : / 20
PAGE No

$$\frac{dz}{dx}\left(\frac{d^2y}{dz^2}\right)+\frac{dy}{dz}\frac{d}{dz}\left(\frac{dz}{dx}\right)\frac{dz}{dx}$$

$$=\frac{dz}{dx}\frac{d^2y}{dz^2}+\frac{dy}{dz}$$

$$=\frac{dz}{dx}\frac{d^2y}{dz^2}+\frac{dy}{dz}\frac{d^2z}{dx^2}\frac{dz}{dx}\frac{dz}{dx}\Big/\frac{dz}{dx}\quad\frac{dx}{dz}$$

$$\frac{d^2y}{dx^2}=\left(\frac{dz}{dx}\right)^2\left\{\frac{d^2y}{dz^2}+\frac{dy}{dz}\cdot\frac{d^2z}{dx^2}\right\}$$

$$\left(\frac{dz}{dx}\right)^2\left\{\frac{d^2y}{dx^2}+\frac{dy}{dz}\cdot\frac{dz^2}{dx^2}\right\}+P\frac{dy}{dz}\cdot\frac{dz}{dx}+Qy=R$$

$$\left(\frac{dz}{dx}\right)^2\left\{\frac{d^2y}{dz^2}+\frac{dy}{dz}\cdot\frac{d^2z}{dx^2}\right\}+P\left(\frac{dy}{dx}\cdot\frac{dz}{dx}\right)+Q\cdot y=R$$

$$\left(\frac{dz}{dx}\right)^2\frac{d^2y}{dx^2}+\left\{\frac{d^2z}{dx^2}\left(\frac{dz}{dx}\right)^2+P\left(\frac{dz}{dx}\right)\frac{dy}{dz}+Qy=R\right.$$

$$\frac{d^2y}{dx^2}+\frac{\left(P\cdot\frac{dz}{dx}+\frac{d^2z}{dx^2}+\left(\frac{dz}{dx}\right)^2\right)\frac{dy}{dz}\cdot+\frac{Q}{\frac{dz}{dx}}y=}{(dz/dx)^2}$$

$$=\frac{R}{(dz/dx)^2}$$

If Q is Perfect square
then method (3) change ind.

DATE : / 20
PAGE No

check u = z
↳ Normal method

$$\left[\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + Q_1 y = R\right]$$

$$\left[P_1 = \frac{P\frac{dz}{dx} + \frac{d^2z}{dx^2}\left(\frac{dz}{dx}\right)^2}{(dz/dx)^2}\right] \times$$

$$Q_1 = \frac{Q}{(dz/dx)^2}, \quad R_1 = \frac{R}{(dz/dx)^2}$$

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$$

Ques:

$$x\frac{d^2y}{dx^2} + (4x^2-1)\frac{dy}{dx} + 4x^3 y = 2x^3$$

$$\frac{d^2y}{dx^2} + \left(\frac{4x^2-1}{x}\right)\frac{dy}{dx} + 4x^2 y = 2x^2 \quad \text{---(1)}$$

$$P = \frac{4x^2-1}{x}, \quad Q = 4x^2, \quad R = 2x^2$$

$$= 4x - \frac{1}{x}$$

DATE : / 20
PAGE No

Q.

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = R_1$$

Q.17

Let $\left(\frac{dz}{dx}\right)^2 = 4x^2 = Q.$

$$\frac{dz}{dx} = 2x \qquad \frac{d^2z}{dx^2} = 2$$

$$z = x^2$$

To change eq^n (1) from Dependent variable x to z

then eq^n (1) can be written as.

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \qquad \text{--(ii)}$$

where $P_1 = \dfrac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \dfrac{2 + \left(4x - \frac{1}{x}\right)2x}{4x^2}$

$= 2$

$$R_1 = \frac{R}{(dz/dx)^2} = \frac{2x^2}{4x^2} = \frac{1}{2}$$

$$Q = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = 1$$

If P_1 = constant
the

DATE: / 20
PAGE No

from eq^n (ii)

$$\frac{d^2y}{dz^2} + 2\frac{dy}{dz} + 1y = \frac{1}{2} \quad \text{—(iii)}$$

A.E

$$(D^2 + 2D + 1)y = \frac{1}{2}$$

$$D = -1, -1.$$

$$C.F = C_1 e^{-x} + C_2 z e^{-x}$$

$$P.I = \frac{\frac{1}{2} e^{0z}}{D^2 + 2D + 1}$$

$$P.I = \frac{1}{2}$$

$$y = C_1 e^{-z} + C_2 z e^{-z} + \frac{1}{2}$$

or $$y = C_1 e^{-x^2} + C_2 x^2 e^{-x} + \frac{1}{2}$$

Q 2. $$\frac{d^2y}{dx^2} - \cot x \frac{dy}{dx} - y \sin^2 x = \cos x - \cos^3 x \quad \text{—(i)}$$

$$P = -\cot x, \quad Q = -\sin^2 x,$$

$$R = \cos x - \cos^3 x$$

DATE : / 20
PAGE No

$$\left(\frac{dz}{dx}\right)^2 = \sin^2 x$$

$$\frac{dz}{dx} = \sin x \qquad \frac{d^2z}{dx^2} = \cos x$$

$$\Rightarrow z = -\cos x$$

Q -

the eq'n can be written as by changing the independent variable x to z.

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$$

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{\cos x + (-\cot x)\sin x}{\sin^2 x}$$

$$P_1 = 0$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = -1$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{\cos x - \cos^3 x}{\sin^2 x}$$

$$= \frac{-z + z^3}{}$$

$$= \frac{\cos x(1 - \cos^2 x)}{\sin^2 x}$$

$$= -z$$

100% Variation of Parameter

DATE: / 20
PAGE No

from (ii)

$$\frac{d^2y}{dz^2} - y = -z.$$

A.E : $m^2 - 1 = 0$

$m = \pm 1$

C.F $= c_1 e^{z} + c_2 e^{-z}$

$$P.I = \frac{-z}{D^2 - 1} = \frac{z}{(1 - D^2)}$$

$$= z(1 - D^2)^{-1}$$

$$= z[1 + D^2]$$

$= z + 0 = z$

P.I $= z$

$y = c_1 e^{z} + c_2 e^{-z} + z$

[Solution: $y = c_1 e^{-\cos x} + c_2 e^{\cos x} - \cos x$]

DATE : / 20
PAGE No

Dmk

Pupi

* Method of variation of Parameters :

Consider a general Second order linear d.e of the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \text{---(ii)}$$

$$\text{(i)} \Rightarrow \frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$$

Let y_1 and y_2 are the linearly independent solution of the eq^n ii) or the corresponding homogenious eq^n (i)

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \quad \text{---(i')}$$

Then

$$y = c_1y_1 + c_2y_2 \quad \text{---(iii)}$$

- where c_1, c_2 are the arbitrary constant is the C.f

⇒

Now we Replace two Constants c_1, c_2 by two Respective functions. then the resultant is given by.

$$y = Ay_1 + By_2 \quad \text{---(4)}$$

Let $y = Ay_1 + By_2$ satisfy the eq^n (i)

DATE: / 20
PAGE No

$$y = Ay_1' + y_1A' + By_2' + B'y_2 \quad -(5)$$

Since we obtain two arbitrary function A and B.

So, we need Require two Condition But we have only one condition

$Ay_1 + By_2$ satisfy eqn (1)

Now to obtain second condition such that first condition not affected

So, Let

$$\boxed{A'By + B'y_2 = 0} \quad -(vi)$$

then

$$y' = Ay_1' + By_2' \quad -(7)$$

$$y'' = Ay_1'' + A'y_1' + By_2'' + B'y_2'$$

Put the values of y_1, y' and y'' in eq(1)

then

$$(Ay_1'' + A'y_1'' + B_2y_2'' + B'y_2') + P(Ay_1' + By_2') + Q(Ay_1 + By_2) = R$$

- CF. $c_1 c_2 \to A, B$
- A & B → obtained

DATE : / 20
PAGE No

$$A(y_1'' + Py_1' + Qy_1) + B(y_2'' + Py_2' + Qy_2) +$$

$$+ A'(y_1') + B'(y_2') = R \quad \text{--- (2)}$$

Since y_1 and y_2 are the linearly independent eq^n of second

then eq^n (8) become :

$$\boxed{A'y_1' + B'y_2' = R} \quad \text{--- (9)}$$

Solving eq^n (VII) and (9) we get the value of A' and B'

$$A' = ? \qquad B' = ?$$

$$A'y_1'y_2 + B'y_2y_2 = y_2 R(x).$$

$$A'y_1y_2' + B'y_1y_2' = 0$$

$$A'(y_1'y_2 - y_1y_2') = y_2 R(x)$$

$$+ A'(y_1y_2' - y_1'y_2) = -y_2 R(x)$$

$$A' = \frac{-y_2 R(x)}{y_1y_2' - y_1'y_2}$$

DATE : / 20
PAGE No

$$A' = \frac{-y_2 R(x)}{W(y_1, y_2)}$$

$$A = -\int \frac{y_2 R(x)}{W(y_1, y_2)} dx + C_1$$

$$B = \int \frac{y_1 R(x)}{W(y_1, y_2)} dx + C_2$$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

where $W(y_1, y_2)$ is known as the Wronskian's function if it is not = 0

Qu: $\frac{d^2y}{dx^2} - 4y = e^{2x}$ — i)

Soln: $(D^2 - 4)y = 0$

$m^2 - 4 = 0$

$m^2 = 4$

$m = \pm 2$

$C.F = C_1 e^{2x} + C_2 e^{-2x}$ — ii)

Let $y = Ae^{2x} + Be^{-2x}$ be the

y_1 y_2 required general sol^n

Let $y = Ay_1 + By_2$

DATE : / 20
PAGE No

where $y_1 = e^{2x}$, $y_2 = e^{-2x}$

$$A = \int \frac{-y_2 R(x)}{y_1 y_2' - y_1' y_2} dx + c_1$$

$$= -\int \frac{e^{-2x} \times e^{2x}}{-2e^{2x} \cdot e^{-2x} - 2e^{-2x} \times e^{2x}} dx + c_2$$

$$= \frac{1}{4}\int dx + c_2 \Rightarrow A = \frac{1}{4}x + c_1$$

$$B = \int \frac{y_1 R(x)}{y_1 y_2' - y_2 y_1'} dx + c_2$$

$$= \int \frac{e^{2x} \cdot e^{2x}}{-4} dx + c_2$$

$$B = -\frac{1}{4}\frac{e^{4x}}{4} + c_2$$

Put the value of A and B in eq (ii)

$$y = \left(\frac{1}{4}x + c_1\right)e^{2x} + \left(-\frac{1}{16}e^{4x} + c_2\right)e^{-2x}$$

$$= c_1 e^{2x} + c_2 e^{-2x} + \frac{1}{4}xe^{2x} - \frac{1}{16}e^{2x}$$

C.F , P.I

Ans.

DATE : / 20
PAGE No

Ques: $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$

$(D^2 - 1) = 0$

C.F = $m^2 - 1 = 0$

$m^2 = 1$

$m = \pm 1$

$C.F = C_1 e^x + C_2 e^{-x}$

Let $y = Ay_1 + By_2$ be the general soln.

$y_1 = e^x, \quad y_2 = e^{-x}$

then

$$A = \int \frac{-y_2 R(x)}{y_1 y_2' - y_2 y_1'} dx + c_1$$

$$= \int \frac{-e^{-x} \times \frac{2}{1+e^x}}{-e^x e^{-x} - e^{-x} e^x} dx + c_1$$

$$= \frac{-2e^{-x}}{1+e^x} = \int \frac{e^{-x}}{1+e^x}$$

$$= \int \frac{e^{-x} e^{-x}}{1+e^{-x}} \Rightarrow \begin{matrix} e^{-x} + t = t \\ -e^{-x} dx = dt \end{matrix} \Rightarrow -\int \frac{(t+1)}{t} dt$$

$$\int \left(\frac{1}{t} - 1\right) dt = \log(e^{-x}+1) - (e^x + 1) + c_1$$

DATE : / 20
PAGE No

$$B = \int \frac{y_1 R(x)}{y_1 y_2' - y_2 y_1'} dx + C_2$$

$$= \int \frac{e^x \cdot \frac{2}{1+e^x}}{-2} dx + C_2$$

$$= -\int \frac{e^x}{1+e^x} dx + C_2$$

$$1 + e^x = t$$

$$e^x dx = dt$$

$$-\int \frac{dt}{t} = -\log(1+e^x) + C_2$$

$y =$

$n(n+1)y \Rightarrow$ Legendre

DATE: / 20
PAGE No

Q. Solve the SODE by Variation of Parameter

$$x^2\frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(1+x)y = x^3.$$

$$\frac{d^2y}{dx^2} - \frac{2}{x}(1+x)\frac{dy}{dx} + \frac{2}{x^2}(1+x)y = x$$

For CF solution

$$\frac{d^2y}{dx^2} - \frac{2(1+x)}{x}\frac{dy}{dx} + \frac{2}{x^2}(1+x)y = 0 \quad \text{---(i)}$$

$$P = -\frac{2}{x}(1+x), \quad Q = \frac{2}{x^2}(1+x)$$

$$R = 0$$

Now, $P + Qx = 0$

then $u = x$.

Let we see the remaning solution of equation

therefore CS of the eq^n (i)

$$y = x \cdot v$$

$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

(x^2-x^2) (x^2-x^2) ↳ Since eq^n (1^2-x^2)
x^2-x^2
0 is $n=0$ so zero order

DATE : / 20
PAGE No

Since $y = xv$ be the c.s of eq^n (iii)
then eq^n (ii) becomes

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{y}\frac{dy}{dx}\right)\frac{dv}{dx} = \frac{R}{y}$$

$$\frac{d^2v}{dx^2} + \left(\frac{-2}{x}(1+x) + \frac{2}{x}\right)\frac{dv}{dx} = 0$$

$$\frac{d^2y}{dx^2} + \left(\frac{+2(1+x)+2}{x}\right)\frac{dv}{dx} = 0$$

Let $\frac{dv}{dx} = z$, $\frac{d^2v}{dx^2} = \frac{dz}{dx}$

$$\frac{dz}{dx} - 2z = 0$$

$$\frac{dz}{z} - 2dx = 0$$

$$\log z = 2x + \log c_1$$

$$\log\frac{z}{c_1} = 2x$$

$$\frac{z}{c_1} = e^{2x}$$

$$z = c_1 e^{2x}$$

DATE : / 20
PAGE No

$$\frac{dv}{dx} = c_1 e^{2x}$$

$$\frac{d^2y}{dx^2}$$

$$v = \frac{c_1}{2} e^{2x} + c_2$$

C.F solution of eq^n (i)

$$y = x\left(\frac{c_1}{2} e^{2x} + c_2\right)$$

$$= x\frac{c_1}{2} e^{2x} + x c_2$$

$$= \frac{c_1}{2} x e^{2x} + c_2 x$$

$\underbrace{\qquad}_{y_1}$ $\underbrace{\qquad}_{y_2}$

Let ~~y be the Complete solution~~ of d.e

$$y = c_1 y_1 + c_2 y_2$$

∴ C.F solution of eq^n (i) is

$$y = \frac{c_1}{2} e^{2x} \cdot x + c_2 x$$

$$y = c x e^{2x} + c_2 x$$

DATE : / 20
PAGE No

(14)

Let $y = Axe^{2x} + Bx$ be the required General sol^n of eq^n (1)

$y_1 = xe^{2x}$, $y_2 = x$.

$$A = \int \frac{-y_2 R(x)}{y_1 y_2' - y_2 y_1'} dx + C$$

$$= -\int \frac{x \cdot x}{xe^{2x} 1 - x[2xe^{2x} + e^{2x}]} dx + C_1$$

$$= -\int \frac{x^2}{xe^{2x} - 2x^2 e^{2x} + xe^{2x}} dx + C$$

$$= +\int \frac{x^2}{2x^2 e^{2x}} = \int \frac{x^1}{2e^{2x}}$$

$$= \frac{1}{2}\int \frac{1}{e^{2x}}$$

$$= \frac{1}{2}\int e^{-2x} \Rightarrow \frac{1}{2}\frac{e^{-2x}}{-2} + C_1$$

$$A = -\frac{1}{4}e^{-2x} + C_1$$ Ans

DATE : / 20
PAGE No

$$B = \int \frac{y_1 R(x)}{y_1 y_2' - y_2 y_1'} dx + C_2$$

$$= \int \frac{x e^{2x} \cdot x}{-2x^2 e^{2x}} dx + C_2$$

$$= -\frac{1}{2} \int dx + C_2$$

$$= -\frac{1}{2} x + C_2$$

$$y = \left(-\frac{1}{4} e^{-2x} + C_1\right) x e^{2x} + \left(-\frac{1}{2} x + C_2\right) x$$

$$= -\frac{1}{4} e^{-2x} \times x e^{2x} + x e^{2x} C_1 + \left(-\frac{1}{2} x^2 + C_2 x\right)$$

$$= x e^{2x} C_1 + C_2 x + \frac{x e^{2x} e^{-2x}}{-4} + \frac{1}{2} x^2$$

$$= x e^{2x} C_1 + C_2 x + \left(\frac{x}{4} + \frac{1}{2} x^2\right)$$ Ans

DATE : / 20
PAGE No

Q. $\frac{d^2y}{dx^2} + (1 - \cot x)\frac{dy}{dx} - y\cot x = \sin^2 x$ — (i)

Ans For C.F solution

$$\frac{d^2y}{dx^2} + (1 - \cot x)\frac{dy}{dx} - y\cot x = 0 \quad \text{— (ii)}$$

$P = 1 - \cot x$, $Q = -\cot x$.

$R = 0$

$$1 - (1 - \cot x) + (-\cot x) =$$

$$1 - x + \cot x - \cot x = 0$$

So, $y = e^{-x} = u$

Let the C.F solution of

Let v be the remaining solution of eq^n (ii)

$\Rightarrow y = u \cdot v$

So, complete solution of eq^n (ii)

$$y = e^{-x} \cdot v \quad \text{— (iii)}$$

DATE: / 20
PAGE No

Now eq^{n} (ii) becomes

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{Q}\frac{dQ}{dx}\right)\frac{dv}{dx} = 0$$

$$\frac{d^2v}{dx^2} + \left(1 - \cot x + \frac{2}{e^x}\cdot(-e^x)\right)\frac{dv}{dx} = 0$$

Let $\frac{dv}{dx} = z$

$$\frac{d^2v}{dx^2} = \frac{dz}{dx}$$

$$\frac{dz}{dx} - (1 + \cot x)z = 0$$

$$\int \frac{dz}{z} = \int (1 + \cot x)\, dx$$

$$\log z = x + \log \sin x + \log c_1$$

$$\log \frac{z}{c_1 \sin x} = x$$

$$z = c_1 \sin x \cdot e^x$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

DATE : / 20
PAGE No

$$\frac{dv}{dx} = c_1 e^x \cdot \sin x$$

$$v = c_1 \int e^x \cdot \sin x + c_2$$

$$v = c_1 \left[\frac{e^x}{2} (\sin x - \cos x) \right] + c_2$$

$$v = \frac{c_1}{2} e^x (\sin x - \cos x) + c_2$$

$$y = e^{-x} \frac{c_1}{2} e^x (\sin x - \cos x) + c_2 e^{-x}$$

$$= \frac{c_1}{2} (\sin x - \cos x) + c_2 e^{-x}$$

let $y = A(\sin x - \cos x) + Be^{-x}$ be the required gen. sol of eq (1)

$y_1 = \sin x - \cos x \qquad y_2 = e^{-x}$

Now $y_1 y_2' - y_2 y_1' =$

$= (\sin x - \cos x)(-e^{-x}) - e^{-x}(\cos x + \sin x)$

$= -e^{-x}\sin x + e^{-x}\cos x - e^{-x}\cos x - e^{-x}\sin x$

$= -2e^{-x}\sin x$

DATE: / 20
PAGE No

$$A = -\int \frac{y_2 R(x)}{y_1 y_2' - y_2 y_1'} dx + C_1$$

$$A = -\int \frac{e^{-x} \sin^2 x}{-2 \sin x e^{-x}}$$

$$= \frac{1}{2} \cos x + C_1$$

$$B = \int \frac{y_1 R(x)}{y_1 y_2' - y_2 y_1'} dx + C_2$$

$$= \int \frac{(\sin x - \cos x) \cdot \sin^2 x}{-2 \sin x e^{-x}} dx + C_2$$

$$= \frac{1}{-2} \int \frac{(\sin x - \cos x) \sin x}{e^{-x}} dx + C_2$$

$$= -\frac{1}{2} \int e^{x} (\sin^2 x - \sin x \cos x) dx + C_2$$

$$-\frac{1}{2} \int e^{x} \left(\frac{1 - \cos 2x}{2} - \frac{1}{2} \sin 2x \right) dx + C$$

DATE : / 20
PAGE No

$$= \frac{1}{4}\int e^x dx + \frac{1}{4}\int e^x \cos 2x\, dx + \frac{1}{4}\int e^x \sin 2x\, dx + c_2$$

Qu $(D^2 + 3D + 2)y = e^{e^x}$ — (I)

C.F $(D^2 + 3D + 2)y = 0$

A.E

$m^2 + 3m + 2 = 0$

$m^2 + m + 2m + 2 = 0$

$m(m+1) + 2(m+1)$

$m = -2, -1$

C.F solution is

$y = c_1 e^{-x} + c_2 e^{-2x}$ — (II)

Let $y = Ay_1 + By_2$ is the general solution of the eq (I)

$y_1 = e^{-x}$, $y_2 = e^{-2x}$

$$A = \int \frac{-y_2 e^{e^x}}{y_1 y_2' - y_2 y_1'} dx + c_1$$

$$B = \int \frac{y_1 e^{e^x}}{y_1 y_2' - y_2 y_1'} dx + c_2$$

DATE: / 20
PAGE No

Ans=

$$y_1y_2' - y_2y_1' = e^{-x}(-2)e^{-2x} - e^{-2x}(-1)e^{-x}$$

$$= -2e^{-x}e^{-2x} + e^{-x}e^{-2x}$$

$$= -e^{-x}e^{-2x} = -e^{-3x}$$

$$A = -\int \frac{e^{-2x}\cdot e^{e^x}}{-e^{-3x}}\,dx + c_1$$

$$= +\int e^x \cdot e^{e^x}\,dx + c_1$$

Let $e^x = t$

$e^x\,dx = dt$

$$= \int e^t\,dt + c_1$$

$$= e^t + c_1$$

$$= e^{e^x} + c_1$$

DATE : / 20
PAGE No

$$B = \int \frac{e^{-x}\, e^{e^x}}{-e^{-3x}}\, dx + C_2$$

$$B = -\int e^{2x} e^{e^x}\, dx + C_2$$

$$B = -\int e^x e^x e^{e^x}\, dx + C_2$$

$e^x = t$

$e^x dx = dt$

$$= -\int t \cdot e^t\, dt + C_2$$

$$B = -e^t(t-1) + C_2$$

Put the value of A and B in eq^n ii

$$= -e^{e^x}(e^x - 1) + C_2$$

General solution of the eq^n (i)

$$y = Ay_1 + By_2$$

$$= (e^{e^x} + C_1)\, e^{-x} + e^{e^x}(e^x - 1)\, e^{-2x}$$

$$= e^{e^x} \cdot e^{-x} + e^{-x} C_1 + e^{e^x} e^{-x} e^{-2x}$$

$$y = C_1 e^{-x} + C_2 e^{-2x} + e^{-x} e^{e^x} - e^{-2x} e^{e^x}(e^x - 1)$$

$$y = C_1 e^{-x} + C_2 e^{-2x} + e^{-x} e^{e^x} - e^{-x} e^{e^x} + e^{-2x} e^{e^x}$$

DATE: / 20
PAGE No

Q. $x^2y'' + xy' - y = x^2e^x$ — (i)

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = e^x$$

For C.F solution

$$x^2y'' + xy' - y = 0 \quad \text{— (ii)}$$

Let $x = e^z \Rightarrow z = \log_e x$

and

$$\frac{d}{dz} = D \quad \text{then eq}^n \text{ (ii)}$$

is can be written as

$$(D(D-1)y + Dy - y) = 0$$

$$(D^2 - D + D - 1)y = 0$$

$$(D^2 - 1)y = 0$$

A.E $\quad m^2 - 1 = 0$

$$m^2 = 1$$

$$m = \pm 1$$

$$C.F = c_1e^x + c_2e^{-x}$$

Complete sol^n of eq^n (ii),

$$y = c_1x + c_2\frac{1}{x}$$

DATE : / 20
PAGE No

~~let $y = Ay_1 + By_2$ is the~~

General solution of eq^n (1)

~~$y_1 = e^x$, $y_2 = e^{-x}$~~

C.f $\Rightarrow$ $y = c_1 x + c_2 \frac{1}{x}$

Let $y = Ay_1 + By_2$ will be the Required General solution of eq^n (1) where

$y_1 = x$, $y_2 = \frac{1}{x}$

$$A = -\int \frac{y_2 R(x)}{y_1 y_2' + y_2 y_1'} dx + c_1$$

$$B = \int \frac{y_1 R(x)}{y_1 y_2' - y_2 y_1'} dx + c_2$$

$y_1 y_2' - y_2 y_1' \Rightarrow x \times \left(\frac{-1}{x^2}\right) - \frac{1}{x} \times 1$

$= \frac{-1}{x} - \frac{1}{x} = -1 - \frac{2}{x}$

$$A = -\int \frac{\frac{1}{x} \cdot x^2 e^x}{-2/x^2} dx + c_1$$

Singular ⇒ undefined at any value

DATE: / 20
PAGE No

$$= \frac{1}{2}\int x^2 e^x + c_2$$

$$A = \frac{1}{2} e^x (x^2 - 2x + 2) + c_2$$

$$B = \int \frac{x \cdot x^2 e^x}{-2} dx + c_2$$

$$-\frac{1}{2}\int x^4 e^x dx + c_2$$

$$-\frac{1}{2}\left[e^x (x^2 + 2x + \right.$$

$$-\frac{1}{2}\left[e^x (x^4 + 4x^3 + 12x^2 + 24x + 24) \right. + c_2$$

Put the value of A and B in equation (ii)

$$y = A y_1 + B y_2$$

DATE : 25/02/2019
PAGE No

27/02/19

* Series solution of the second order homogenious linear d.e :-

An expression of the form

$$a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 - - -$$

$$- - - + a_n(x-x_0)^n + - - -$$

Or

$$\sum_{r=0}^{\infty} a_r (x-x_0)^r$$ In terms of $(x-x_0)$ and about the

It is also known as the Power series about the point x_0 where $a_0, a_1, a_2 - - - a_n$ are the constant coefficient of the Power series.

If $x_0 = 0$ then above series can be written as

$$a_0 + a_1 x + a_2 x^2 + - - - a_r x^r + - -$$

or

$$\sum_{r=0}^{\infty} a_r x^r$$

DATE : / 20
PAGE No

* Ordinary point and singular point :-

Consider a diff. eq^{n}

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0 \quad \text{--- (i)}$$

where $P(x)$, $Q(x)$, $R(x)$ are the polynomial in term of x

If the value of $P(x)$ does not zero for the point $x=a$,
then $x=a$ is the Ordinary point of the equation (i)

Otherwise Point $x=a$ is the singular point of the d.E (i)

Note :- i) If $x=a$ is an Ordinary point of the given d.E (i) then we assume the series solution of the eq^{n} (i) is

$$y = \sum_{r=0}^{\infty} a_r x^r$$

ii) If $x=a$ is the singular point of eq^{n} (i)

DATE : / 20
PAGE No

The we assume the sol^{n} of the d.E is of the form

$$y = x^m \sum_{r=0}^{\infty} a_r x^r$$

$$y = \sum_{r=0}^{\infty} a_r x^{r+m}$$

Legendre eq^{n}

Q $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - x.y = 0$ — (i)

Sol^{n} Since $x=0$ is an ordinary point of the d.E i.e. so we assume the solution of the eq^{n} (i),

$$y = \sum_{r=0}^{\infty} a_r x^r \quad \text{— (ii)}$$

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r . r x^{r-1} \quad \text{— (iii)}$$

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r r(r-1) x^{r-2} \quad \text{— (iv)}$$

Put the value of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in eq^{n} (i)

DATE: / 20
PAGE No

$$(1+x^2)\sum_{r=0}^{\infty} a_r r(r-1)x^{r-2} + x\left(\sum_{r=0}^{\infty} a_r r\, x^{r-1}\right) - x\left(\sum_{r=0}^{\infty} a_r x^r\right) = 0$$

$$\sum_{r=2}^{\infty} a_r r(r)$$

$$\sum_{r=2}^{\infty} a_r r(r-1)x^{r-2} + \sum_{r=0}^{\infty} a_r r(r-1)x^r +$$

$$+\sum_{r=0}^{\infty} a_r r(x)^r - \sum_{r=0}^{\infty} a_r x^{r+1} = 0$$

$x=0$ $\quad a_2(2)(2-1) + a_0 \cdot 0 + a_0 \cdot 0 = 0$

$$\boxed{a_2 = 0}$$

$x=1$ $\quad a_3\, 3(3-1) + a_1(1-1) + a_1 \cdot 1 = a_3$

$$6a_3 = a_0 - a_1$$

$$a_3 = \frac{a_0 - a_1}{6}$$

$x=2$ $\quad a_4\, 12 + a_2\, 2(2-1) + a_2 2 - a_1 = 0$

$$12a_4 + 2a_2 + 2a_2 - a_1 = 0$$

DATE : / 20
PAGE No

$12a_4 - a_1 = 0$

$$\boxed{a_4 = \frac{a_1}{12}}$$

$$y = \sum_{r=0}^{\infty} a_r x^r$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$y = a_0 + a_1 x + 0 \cdot x^2 + \left(\frac{a_0 - a_1}{6}\right) x^3 + \frac{1}{12} a_1 x^4 + \cdots$$

$$\boxed{y = a_0\left(1 + \frac{x^3}{6} + \cdots\right) + a_1\left(x - \frac{x^3}{6} + \frac{x^4}{12} + \cdots\right)}$$

↳ Ans

DATE: / 20
PAGE No

Ques solve in series the legendre's eq^n.

Bessel eqn → $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + p(p+1)y = 0$ — i)

Sol^n. Since coefficient of $\frac{d^2y}{dx^2}$ is not at $x=0$, then $x=0$ is an ordinary point of eq^n (i) then solution of eq^n (i),

$$y = \sum_{r=0}^{\infty} a_r x^r \quad \text{— ii)}$$

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r \cdot r \cdot x^{r-1} \quad \text{— (iii)}$$

also

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r r(r-1) x^{r-2} \quad \text{— iv)}$$

Putting the value of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in equation i)

$$(1-x^2)\sum_{r=0}^{\infty} a_r r(r-1)x^{r-2} - 2x\sum_{r=0}^{\infty} a_r r.x^{r-1} + p(p+1)\sum_{r=0}^{\infty} a_r x^r =$$

$$\sum_{r=0}^{\infty} a_r r(r-1)x^{r-2} - \sum_{r=0}^{\infty} a_r r(r-1)x^r - 2\sum_{r=0}^{\infty} a_r r x^r + p(p+1)\sum_{r=0}^{\infty} a_r x^r =$$

$r^2+p^2-p-p+(2r+p)=$

DATE : / 20
PAGE No

$$\sum_{r=0}^{\infty} a_r r(r+1)x^{r-2} - \sum_{r=0}^{\infty} a_r\left(r^2-r+2r-p(p+1)\right)x^r = 0$$

$$\sum_{r=0}^{\infty} a_r(r)(r+1)\,x^{r-2} - \sum_{r=0}^{\infty} a_r\left(r^2+r-p^2-p\right)x^r = 0$$

$$\sum_{r=0}^{\infty} a_r r(r+1)x^{r-2} = \sum_{r=0}^{\infty} a_r (r-p)(r+p+1)$$

taking the

$$a_{r+2}(r+2)(r+1) = a_r (r-p)(r+p+1)$$

$$a_{r+2} = \frac{(r-p)(r+p+1)}{(r+1)(r+2)} a_r \quad \text{---(iii)}$$

→ Put the value of $r = 0, 1, 2, 3, 4$ in the recreance Relation (iii)

$$a_2 = \frac{-p(p+1)}{2} a_0 = \frac{-p(p+1)}{2!} a_0 =$$

$$a_3 = \frac{(1-p)(p+2)}{2\times 3} a_1 = \frac{-(p-1)(p+2)}{3!} a_1$$

$$a_4 = \frac{(2-p)(p+3)}{3\times 4} a_2 = \frac{p(p+1)(p-2)(p+3)}{4!} a_0$$

$$a_5 = \frac{(3-p)(p+4)}{4\times 5} a_3 = \frac{+(p-3)(p+4)(p+1)(p+2)}{20} \frac{}{3!} a_1$$

$$= \frac{(p-3)(p-1)(p+2)(p+4)}{5!} a_1$$

DATE : / 20
PAGE No

$$a_6 = \frac{(4-p)(p+5)}{5\times 6} a_4$$

$$= \frac{-(p-4)(p+5)}{30} \times \frac{p(p+1)(p-2)(p+3)}{4!}$$

$$a_6 = \frac{-(p-4)p(p-4)(p+5)(p+1)(p-2)(p+3)}{6!} a_0$$

Put the value of $a_2, a_3, a_4 \cdots$ in eq^{n} (ii)

$$y = \sum_{r=0}^{\infty} a_r x^r$$

$$= \sum_{r=0}^{\infty} a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$\Rightarrow a_0 + a_1 x - \frac{p(p+1)}{2!} x^2 a_0 - \frac{(p-1)(p+2)}{3!} a_1 x^3 +$$

$$+ \frac{p(p+1)(p-2)(p+3)}{4!} x^4 a_0 + \frac{(p-3)(p-1)(p+2)}{5!} \frac{(p+1)}{a_1} x$$

$$+ \frac{-p(p-4)(p+5)(p-2)(p+1)(p+3)}{6!} a_0 x^6$$

$$y \Rightarrow a_0 \left\{ 1 - \frac{p(p+1)}{2!} x^2 + \frac{p(p+1)(p-2)(p+3)}{4!} x^4 + \cdots \right.$$

$$\left. - \frac{p(p-4)(p-2)(p+3)(p+1)(p+5)}{6!} x^6 + \cdots \right]$$

DATE : / 20
PAGE No

$$+a_1\left\{x - \frac{(p+1)(p+2)}{3!}x^3 + \frac{(p-3)(p+1)(p+3)(p+4)x^5}{5!}\right.$$

$$\left. + \text{---} \quad \text{---} \quad +\cdots \infty \right\}$$

Qu:2: Solve:

$y'' + (x+1)^2 y' - (4(n+1))y = 0$ in series about the Point $x = 1$ ——— (i)

Sol: Let $x + 1 = z$

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}$$

$$\frac{dy}{dx} = \frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{x}\left(\frac{dy}{dz}\right)$$

$$= \frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}$$

Now eq^4 (i) can be written as

$$\frac{d^2y}{dz^2} + z^2\frac{dy}{dz} - 4zy = 0 \quad \text{——— (ii)}$$

DATE: / 20
PAGE No

$$y = \sum_{r=0}^{\infty} a_r z^r \quad \text{---(i)}$$

$$\frac{dy}{dz} = \sum_{r=0}^{\infty} a_r r z^{r-1}$$

$$\frac{d^2y}{dz^2} = \sum_{r=0}^{\infty} a_0 r(r-1) z^{r-2}$$

Putting the value of y, $\frac{dy}{dz}$, and $\frac{d^2y}{dz^2}$ in equation (i)

$$\sum_{r=0}^{\infty} a_r r(r-1) z^{r-2} + z^2 \sum_{r=0}^{\infty} a_r r z^{r-1} - 4z \sum_{r=0}^{\infty} a_r z^r = 0$$

$$\sum_{r=0}^{\infty} a_r r(r-1) z^{r-2} + \sum_{r=0}^{\infty} a_r r z^{r+1} - 4 \sum_{r=0}^{\infty} a_r z^{r+1} = 0$$

$$\sum_{r=0}^{\infty} a_r r(r-1) z^{r-2} + \sum_{r=0}^{\infty} a_r z^{r+1}(r-4) = 0 \quad \text{---(ii)}$$

Equating to zero, the coefficient of z^{r+1}

$$a_{r+3}(r+3)(r+2) + a_r(r-4) = 0$$

$$a_{r+3} = \frac{(4-r)}{(r+3)(r+2)} a_r \quad \text{---(iii)}$$

DATE : / 20
PAGE No

again equate to zero to the constant in the identity (2)

$$a_2\, 2(1) = 0$$

$$\boxed{a_2 = 0}$$

Put $r = 0, 1, 2, 3, 4, \ldots$ in the relation (V)

$$a_3 = \frac{(r=0)(4-0)}{3 \cdot 2} a_0 = \frac{4}{6} a_0 = \frac{2}{3} a_0$$

$$a_4 = \frac{3}{4 \cdot 3} a_1 = \frac{3}{12} a_1 = \frac{1}{4} a_1$$

$$a_5 = \frac{2}{5 \cdot 4} a_2 = 0$$

$$a_6 = \frac{1}{45} a_0$$

$$= y = \sum_{r=0}^{\infty} a_r x z^{r+1}$$

$$= \sum_{r=0}^{\infty} a_1 r$$

$$y = \sum_{r=0}^{\infty} a_r z^r$$

$$= a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5$$

DATE: / 20
PAGE No

* Solution near a Singular Point :-

In this case we assume the solution of the d.e is of the form

$$y = x^m \sum_{r=0}^{\infty} a_r x^r$$

where $a_0 \neq 0$

assume solution $\rightarrow$ $$y = \sum_{r=0}^{\infty} a_r x^{r+m} \quad \text{---(1)}$$

- find dy/dx
- find d^2y/dx^2
- Put the value of y, dy/dx, d^2y/dx^2 in given diff. eqn.
- and obtained an identity in terms of x
- by equating to zero the coefficient of lowest power of x.
- get quadratic eq^n in terms of m. It is called the Indicial equation
- To determine the value of m
- equating to zero the coefficients of other various powers of x. and we can obtained the values of constants a_1, a_2, a_3 ---- in term of a_0.

DATE : / 20
PAGE No

The following cases arise acc. to the nature of the roots of the indicial equation

Case-1. Roots of the equation is equal
C-2. Roots of the indical eq^n is unequal and different by a quantity not an integer.
C-3. Roots of the indicial eq^n is unequal and different by an integer and making the coefficients of some powers of x in the series for y infinity,
C-4 Roots of the Indicial equation unequal differening by an integer and making the same power of x in the series for y indeterminate

Case-1 Roots of the indicial equation is equal
($m_1 = m_2$)

Q Solve the d.e

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0 \quad \text{---} \quad (i)$$

Coefficient of d^2y/dx^2 is zero at the point $x=0$. Therefore $x=0$ is a singular point of the d.eq^n. (i)

DATE: / 20
PAGE No

then let us assume the solution of the eq ①

$$y = x^m \sum_{r=0}^{\infty} a_r x^r \quad ; \quad a_0 \neq 0$$

$$y = \sum_{r=0}^{\infty} a_r x^{r+m}$$

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (r+m) x^{r+m-1}$$

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (r+m)(r+m-1) x^{r+m-2}$$

putting the value of eq $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in eq ①

$$x \sum_{r=0}^{\infty} a_r (r+m)(r+m-1) x^{r+m-2} + \sum_{r=0}^{\infty} a_r (r+m) x^{(r+m-1)} + x \sum_{r=0}^{\infty} a_r x^{r+m} = 0$$

$$\sum_{r=0}^{a} a_r (r+m)(r+m-1) x^{r+m-1} + \sum_{r=0}^{\infty} a_r (r+m) x^{(r+m-1)} + \sum_{r=0}^{\infty} a_r (x)^{r+m+1} = 0$$

DATE: / 20
PAGE No

$$\sum_{r=0}^{\infty} a_r (r+m) x^{r+m+1} (r+m+1) + \sum_{r=0}^{\infty} a_r (x)^{r+m+1} = 0 \quad \text{---(ii)}$$

$$\sum_{r=0}^{\infty} a_r (r+m)^2 x^{r+m+1} + \sum_{r=0}^{\infty} a_r (x)^{r+m+1} = 0$$

equating to zero the lowest term of x i.e x^{m+1}

$$a_0 (m)^2 x^{m+1} = 0 \qquad a_0 \neq 0$$

$$\boxed{m^2 = 0 \Rightarrow m = 0, 0}$$

Again equate to zero the coefficient of x^{r+m+1}

$$a_{r+2}(r+m+2)^2 + a_r = 0$$

$$a_{r+2} = \frac{-a_r}{(r+m+2)^2}$$

To obtain the value of a_1 equate to zero the coefficient of x^m

DATE : / 20
PAGE No

$$a_1(1+m)^2 = 0$$

$$\boxed{a_1 = 0}$$

Put $r = 0, 1, 2, 3 \ldots$ in recurrence

$$a_2 = \frac{-a_0}{(m+2)^2} = \frac{-a_0}{4}$$

$$a_3 = -0$$

$$a_4 = \frac{-a_2}{(m+4)^2} = \frac{+a_0}{(m+2)^2(m+4)^2}$$

Sol: $y = x^m(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots \infty)$

$$= x^m\left(a_0 + 0 - \frac{a_0}{(m+2)^2}x^2 + 0 \cdot x^3 + \right.$$

$$\left. \frac{a_0}{(m+2)^2(m+4)^2}x^4 + \ldots \infty\right)$$

$$x^m a_0\left(1 - \frac{1}{(m+2)^2}x^2 + \frac{1}{(m+2)^2(m+4)^2}x^4 + \ldots\right)$$

DATE: / / 20
PAGE No

Put $m=0$

$$y = a_0\left(1 - \frac{1}{(m+2)^2}x^2 + \frac{1}{(m+2)^2(m+4)^2}x^4 + \cdots \infty\right) \quad \text{(v)}$$

$$y = a_0\left(1 - \frac{1}{2^2}x^2 + \frac{1}{2^2 4^2}x^4 + \cdots \infty\right)$$

$$y = a_0 y_1 \Rightarrow a y_1 \quad (\text{replace } a_0 = a)$$

Since y_1 is a linearly independent sol^n of eq^4 (i) at $m=0$. To obtain the second independent solution of eq^4 (i) at $m=0$ ∞, Partially differentiate eq^4 (v) w.r.t m.

$$\frac{\partial y}{\partial m} = a_0 x^m \left\{0 + \frac{2}{(m+2)^3} - 2\left(\frac{1}{(m+2)^3(m+4)^3} + \frac{1}{(m+2)^3(m+4)^2}\right)x^4 + \cdots \infty\right\} + a_0 x^m \log x \left\{1 - \frac{1}{(m+2)^2}x^2 + \frac{1}{(m+2)^2(m+4)^2}x^4 + \cdots \infty\right\}$$

Put $m=0$

$$\left(\frac{dy}{dm}\right)_{m=0} = a_0\left[\frac{2x^2}{8} - 2\left\{\frac{1}{2^2 4^3} + \frac{1}{2^3 4^2}\right\}x^4 + \cdots\right] + a_0 \log_e x \left\{1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} + \cdots + \infty\right\}$$

DATE : / 20
PAGE No

16 × 8 = 128

$$= a_0\left\{\log_e x\left\{1-\frac{x^2}{2^2}+\frac{x^4}{2^2 4^2}+\cdots\right\}+\frac{1}{4}+\frac{1}{4.64}+\frac{1}{8.16}\right.$$

$$\left.\left\{\frac{x^2}{2^2}-\frac{3}{2.4^3}x^4+\cdots\right\}\right\}$$

Replace $a_0 = b$

$$\left(\frac{\partial m}{\partial y}\right)_{m=0} = by_2$$

Hence Complete series solution of eq^n (1)

$$y = ay_1 + by_2$$

where a and b are two arbitrary constant.

Qu: Solve differential equation

$$(2x+x^3)\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6xy = 0 \quad \text{---(1)}$$

obtain the series sol. about $x=0$

Sol^n Since $x=$ Coefficient of y'' is zero at $x=0$

so, $x=0$ is an singular Point of the eq^n (1)

Let us assume the sol^n of eq^n (1)

DATE: / 20
PAGE No

$$y = x^m \sum_{r=0}^{\infty} a_r x^r, \quad a_0 \neq 0 \quad \text{--- (1)}$$

$$= \sum_{r=0}^{\infty} a_r x^{r+m}$$

$$y' = \sum_{r=0}^{\infty} a_r (r+m) x^{(r+m-1)}$$

$$y'' = \sum_{r=0}^{\infty} a_r (r+m)(r+m-1) x^{(r+m-2)}$$

· Put the value of y, y' and y'' in eq (1)

$$\Rightarrow (2x+x^3) \sum_{r=0}^{\infty} a_r (r+m)(r+m-1) x^{(r+m-2)} -$$

$$- \sum_{r=0}^{\infty} a_r (r+m) x^{(r+m-1)} - 6x \sum_{r=0}^{\infty} a_r x^{r+m} = 0$$

$$\Rightarrow 2 \sum_{r=0}^{\infty} a_r (r+m)(r+m-1) x^{(r+m-1)} + \sum_{r=0}^{\infty} a_r (r+m)(r+m-1) x^{(r+m+1)}$$

$$- \sum_{r=0}^{\infty} a_r (r+m) x^{(r+m-1)} - 6 \sum_{r=0}^{\infty} a_r x^{r+m+1} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r \left[2(r+m)(r+m-1) - (r+m) \right] x^{(r+m-1)}$$

$$+ \sum_{r=0}^{\infty} a_r \left\{ (r+m)(r+m-1) - 6 \right\} x^{r+m+1} = 0$$

DATE : / 20
PAGE No

$$\sum_{r=0}^{\infty} a_r \{ 2(r+m)^2 - 3(r+m) \} x^{r+m+1}$$

$$+ \sum_{r=0}^{\infty} a_r \{ (r+m)^2 - (r+m) - 6 \} x^{r+m+1} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r (r+m)(2r+2m-3) x^{r+m+1} +$$

$$\sum_{r=0}^{\infty} a_r [r+m-3)(r+m+2) x^{r+m+1} = 0 \quad \leftarrow (iii)$$

by equating the to zero the coefficient of lowest degree i.e x^{m+1} to obtain the Indicial eq. for

$$a_0 (m)(2m-3) = 0 \qquad a_0 \neq 0$$

$m(2m-3)=0 \Rightarrow m=0$

$m = 3/2$

Roots of the Indicial equation are not equal and not differing by an integer.

Again equating to zero the coefficient of x^{r+m+1} to obtain the Recreance Relation

Legendre or Bessel Polynomial

DATE: / / 20

PAGE No

$a_{r+2}(r+m+2)(2r+2)$

$a_{r+2}(r+m+2)(2r+2m+1) + a_r(r+m-3)(r+m+2) = 0$

$a_{r+2} = -\frac{a_r(r+m-3)(r+m+2)}{(r+m+2)(2r+2m+1)}$

$$\boxed{a_{r+2} = -\frac{(r+m-3)}{(2r+2m+1)} a_r} \quad \text{(IV)}$$

To obtain the value of a_1 equating to zero, coefficient of x^m.

$a_1(1+m)(2m-2) = 0$

$\boxed{a_1 = 0}$ { for both the value of m.

by recreance Relation

$a_3 = 0$, $a_5 = 0$