Matrix

21/08/18

* Transpose Conjugate of a matrix :- (A^θ)

The matrix is obtain after taking transpose of a conjugate matrix is known as transpose conjugate of a matrix.

Ex:-

$$A = \begin{bmatrix} 2+i & 3 & i \\ 4-i & 8 & 8i \\ 9 & -4 & 3+i \end{bmatrix}$$

$$A^\theta = \begin{bmatrix} 2-i & 4+i & 9 \\ 3 & 8 & -4 \\ -i & -8i & 3-i \end{bmatrix}$$

* Unitry Matrix :-

A matrix A is said to be unitry matrix

if $\boxed{A^\theta A = I}$

* Hermitian matrix :-

if $\boxed{A = A^\theta}$

* Skew-Hermitian matrix :-

if $\boxed{A^{\theta} = -A}$

The Principle diagonal elements of Skew Hermitian matrix either Purely imaginery or zero.

Ex :-

$$A = \begin{bmatrix} -i & +3+4i \\ -3+4i & 0 \end{bmatrix}$$

$$A^{\theta} = \begin{bmatrix} i & -3-4i \\ 3-4i & 0 \end{bmatrix} = -A$$

* Idompoment Matrix :-

if $\boxed{A^2 = A}$

* Periodic matrix :-

if $\boxed{A^{K+1} = A}$

where K is Positive intiger if K is a least Positive intiger for which $A^{K+1} = A$ then K is called Periodi of A

* Nelpotent Matrix :-

if $A^K = 0$

↳ Null Matrix

where K is positive intiger.
if K is the least positive intiger for which

$A^K = 0$ then K is called index of A

$A^2 \neq 0$ $A^4 = 0$ then all equal zero
$A^3 \neq 0$ $A^5 = 0$

- B will be called inverse of A
if $AB = BA = I$

Que:- find the inverse of a matrix by elementry transformation.

Sol^n :-

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

$A = IA$

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 + 2R_1$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & -6 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$

$R_2 \to \frac{R_2}{2}$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -3 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1/2 & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 - R_2$
$R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & -3 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 3/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 1 & 1 & 1 \end{bmatrix} A$$

$R_3 \to \frac{R_3}{-4}$

$$\begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ -1/4 & -1/4 & -1/4 \end{bmatrix} A$$

$R_1 \to R_1 - 6R_3$
$R_2 \to R_2 + 3R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix} A$$

(b) $$A^{-1} = \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$$

* Minor :-

If A be any m×n matrix then the det. of every square sub-matrix of A is called minors of matrix A.

* Rank of Matrix :-

• Rank :- A matrix is said to of Rank r if

(i) it has at least one non-zero minor of order r.

(ii) All the minors of order highest then r must be zero

• ② Normal form :-

Every non zero A matrix of rank r can be reduced by a sequence of elementry transformations to the form.

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} I_r & 0 \end{bmatrix} \text{ or } \begin{bmatrix} I_r \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} I_r \end{bmatrix}$$

Ex:- $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -2 & 4 \end{bmatrix}$ find Rank.

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$ $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$C_2 \to C_2 - 2C_1$, $C_3 \to C_3 - 3C_1$, $C_4 \to C_4 - 4C_1$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$R_3 \to R_2$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C_2 \leftrightarrow C_3$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$

$\text{Rank}(A) = 2$.

Q. $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$R_1 \leftrightarrow R_2$ $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$R_3 \to R_3 - 3R_1$
$R_4 \to R_4 - R_1$

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$C_4 \to C_4 - C_1$
$C_3 \to C_3 - C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$
$R_4 \to R_4 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \to C_4 + C_2$
$C_3 \to C_3 + 3C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

(Q) Rank = 2

(iii) Echelon form :-

A matrix A is said to be an echelon form if its
(i) Every row of A which has all entries zero must lies in the last.
(ii) the no. of zeros before the first non-zero elements in a row is less then the no. of such zeroes in the next row. the rank of the matrix A is the no. of non-zero rows in the echelon form of the matrix.

Ex:

$$\begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$
$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & 3 & -9 & -3 \\ 0 & -2 & 4 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_3 \to R_3 + R_2 \to R_2/3$

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & -2 & 4 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_3 \to R_3 + 2R_1$
$R_4 \to R_4 - R_2$

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

$R_3 \to R_3/-2$

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

$R_4 \to R_4 - 4R_3$

$$\begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank $[A] = 3$

Q. $$A = \begin{bmatrix} 1 & -2 & 2 & 4 \\ -2 & 4 & -1 & 3 \\ -1 & 2 & 7 & 6 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$, $R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & -2 & 2 & 4 \\ 0 & 0 & 3 & 11 \\ 0 & 0 & 9 & 10 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$\begin{bmatrix} 1 & -2 & 2 & 4 \\ 0 & 0 & 3 & 11 \\ 0 & 0 & 0 & -23 \end{bmatrix}$$

Rank(A) = 3

Q) 10

* Linear Simaltinious equation :

Q. Solve the Simaltinious equations :

$$2x + 3y + 3z = 5$$
$$x - 2y + z = -4$$
$$3x - y - 2z = 3$$

$$AX = B$$

$$\begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$R_2 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & 3 \\ 3 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 5 \\ 3 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 7 & 1 \\ 0 & 5 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 13 \\ 15 \end{bmatrix}$$

~~Q2 →C~~
~~$R_2 \to R_2 + 3R_1$~~
$R_2 \to R_2/7$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 1/7 \\ 0 & 5 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 13/7 \\ 15 \end{bmatrix}$$

$R_3 \to R_3 - 5R_2$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 1/7 \\ 0 & 0 & -40/7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 13/7 \\ 40/7 \end{bmatrix}$$

$x - 2y + z = -4$

$y + \frac{z}{7} = 13/7$

$-4.0z = \frac{4.6}{7}$, $z = -1$

$y + \frac{-1}{7} = \frac{13}{7}$

$y = \frac{14}{7} = 2$, $x - 4 + (-1) = -4$

$x - 5 = -4$

$x = 1$

* Types of Linear equations :

(i) Consistent eq^n :

* System of eq^n is said to be consistent if they have one or more solution.

(ii) Inconsistent eq^n :

If the system eq^n have no solution then it is said to be inconsitent eq^n

$$a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1$$
$$a_{21}x_2 + a_{22}x_2 + \cdots a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = b_m$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \text{Coff. matrix}$$

$$C = \left[\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

↳ Augmented matrix

→ Consistent eq^n:

if Rank (A) = Rank (C)

(i) If Rank (A) = Rank (C) = no. of unknowns
Unique solution.

(ii) if Rank (A) = Ran (C) < no. of unknowns
Infinite no. of sol^n.

→ Inconsistent eq^{n} :

if Rank(A) ≠ Rank(C)

Q. Check weather the given system of equation are consistent ~~and~~ or Inconsistent

$x+y+z = -3$

$3x+y-2z = -2$

$2x+4y+7z = 7$

Sol^{n}

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \\ 2 & 4 & 7 \end{bmatrix}$$

$$C = \left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 3 & 1 & -2 & +2 \\ 2 & 4 & 7 & 7 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 2R_1$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 2 & 5 & 13 \end{array}\right]$$

~~$R_2 \to R_2$~~

$R_3 \to R_3 + R_2$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 0 & 0 & 20 \end{array}\right]$$

Rank(C) = 3

Rank(A) = 2

Rank(C) ≠ Rank(A)

Inconsistent.

Q. For what value of λ will the following eqⁿ fail to have unique sol^{n}.

$$3x - y + \lambda z = 1$$
$$2x + y + z = 2$$
$$x + 2y - \lambda z = -1$$

Sol^{n}:

$$C = \left[\begin{array}{ccc:c} 3 & -1 & \lambda & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 2 & -\lambda & -1 \end{array}\right]$$

$R_1 \leftrightarrow R_3$

$$\left[\begin{array}{ccc:c} 1 & 2 & -\lambda & -1 \\ 2 & 1 & 1 & 2 \\ 3 & -1 & \lambda & 1 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$

$$\left[\begin{array}{ccc:c} 1 & 2 & -\lambda & -1 \\ 0 & -3 & 1+2\lambda & 4 \\ 0 & -7 & 4\lambda & 4 \end{array}\right]$$

$R_2 \to R_2/3$

$$\left[\begin{array}{ccc:c} 1 & 2 & -\lambda & -1 \\ 0 & 1 & \frac{-1+2\lambda}{3} & -4/3 \\ 0 & -7 & 4\lambda & 4 \end{array}\right]$$

$4\lambda - \frac{7+14\lambda}{3}$

$\frac{12\lambda - 14\lambda - 7}{3}$

$\frac{-2\lambda - 7}{3}$

$R_3 \to R_3 + 7R_2$

$$\left[\begin{array}{ccc:c} 1 & 2 & -\lambda & -1 \\ 0 & 1 & \frac{-1+2\lambda}{3} & -4/3 \\ 0 & 0 & -\frac{7-2\lambda}{3} & -16/3 \end{array}\right]$$

Rank $(A) = 3$

Rank $(A) = 3$ if $\frac{-7-2\lambda}{3} \neq 0$

$$\frac{-7-2\lambda}{3} \neq 0$$

$$\lambda \neq -7/2$$

Q. For what value of η the system of eq^{n}s

$$x+y+z=1$$
$$x+2y+4z=\eta$$
$$x+4y+10z=\eta^2$$

has (i) unique solution.
(ii) infinite many sol^{n}
(iii) no solution

$$C=\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \eta \\ 1 & 4 & 10 & \eta^2 \end{array}\right]$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \eta-1 \\ 0 & 3 & 9 & \eta^2-1 \end{array}\right]$$

$R_3 \to R_3 - 3R_2$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \eta-1 \\ 0 & 0 & 0 & \eta^2-3\eta+2 \\ & & & =K \end{array}\right]$$

Rank(A) = 1 or 2
Rank(C) = 1 or 2

(i) if $K=0$ or $\eta=1$ or $\eta=2$

Rank(A) = Rank(C) < no. of unknown

Then their exist infinite many sol^n

(ii) if $K \neq 0$ $\eta \neq 1$ or $\eta \neq 2$

$$\text{Rank}(A) \neq \text{Rank}(C)$$

their Exist no sol^n. the give system

(iii) their exist no value of η for which eq^n have unique solution.

for $\eta = 1$

$$C = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

also find in each case

Rank(C) = 2, no. of var. = 3

$(n-r) = 1$ variables must have const. value

$x+y+z=1$

$y+3z=0$

let $z=c$

$y=-3$

$x = 1+2c$, $c \neq 0$

If $n=2$

$$C=\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 3 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$n-r=1$

$x+y+z=1$

$y+3z=1$

$z=c_1$

$y=1-3c_1$

$x=2c_1$, $c\neq 0$

Que :- (A) For what value of λ and μ the given simultaneous eq^n

$x+y+z=6$

$x+2y+3z=10$

$x+2y+\lambda z=\mu$.

have

(i) no solution

(ii) unique "

(iii) infinite sol^n.

Q. Test the consistency and solve it.

$$x - 2y - z = 5$$
$$x + 8y - 3z = -1$$
$$2x + y - 3z = 7$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 5 \\ 1 & 8 & -3 & -1 \\ 2 & 1 & -3 & 7 \end{array}\right]$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 5 \\ 0 & 10 & -2 & -6 \\ 0 & 5 & -1 & -3 \end{array}\right]$$

$R_3 \to R_3 - \frac{R_2}{2}$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 5 \\ 0 & 10 & -2 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$r = 2, \quad n = 3$

$n - r = 1$ variables has arbitary constant value

$$x - 2y - z = 5$$
$$10y - 2z = -6 \qquad \text{let } z = c$$

$$y = \frac{-6 + 2c}{10}$$

$$x = 5 - \frac{2}{10}(-6 + 2c) = -\frac{1}{5}(-6 + 2c)$$

$$x = \frac{7c + 19}{5} \qquad = -\frac{2c + 6}{5}$$

* Eigen values & Eigen vectors :

If $A = [a_{ij}]_{n \times n}$ square matrix then matrix $[A - \lambda I]$ is called characteristic matrix of A.

Also the determinant $|A - \lambda I|$

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix}$$

is a Polynomial eqn in λ of degree n

Hence $|A - \lambda I| = 0$ is a characteristic eqn of A of degree n.

clearly after solving this eqn we get n values of λ these n values are called eigen values or characteristic value of A.

⇒ The set of all the eigen value of A is called spectrum of A.

* Properties :

(1) The eigen value of A and its transpose A' are same

(ii) the sum of eigen values of A is always equal to trace of A

trace (= Sum of principle diagonal elem. of A)

(iii) the Product of eigen values of A must be equal to determinant of A.

(iv) If $\lambda_1, \lambda_2, \lambda_3 \dots \lambda_n$ are the eigen value of A then

KA are $K\lambda_1, K\lambda_2 \dots K\lambda_n$

A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2} \dots \frac{1}{\lambda_n}$

A^m are $\lambda_1^m, \lambda_2^m \dots \lambda_n^m$

Q. Find the eigen values of matrix A

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$ and find out $A^3 - 5A^2 + 8A - I$

The char. eq^n of A is

$|A - \lambda I| = 0$

$2(4) - 2(1) + 1($

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix}$$

8 - 2 - 1 = 5

$= -2-\lambda((3-\lambda)(2-\lambda))$

$\lambda^3 - (\text{trace of } A)\lambda^2 + (C_{11}+C_{22}+C_{33})\lambda - \det(A) = 0$

$\lambda^3 - 7\lambda^2 + (4+3+4)\lambda - 5 = 0$

$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

$\lambda^2(\lambda-1) - 6\lambda(\lambda-1) + 5(\lambda-1) = 0$

$\lambda = 1, \quad \lambda^2 - 6\lambda + 5 = 0$

$\lambda = 1, 1, 5$

Eigen value of A^2 are 1, 1, 25

" " " A^3 " 1, 1, 125

Eigen value of $A^3 - 5A^2 + 2A - I$

are $1 - (5\times1) + 2(1) - 1 = -3$

are $1 - 5 + 2 - 1 = -3$

are $125 - 125 + 10 - 1 = 9$

* Eigen Vectors :-

If λ is any eigen value of A then their exist a non-zero vector X such that

$$AX = \lambda X$$

$$[A - \lambda I]X = 0$$

then X is called eigen vector of A.

Corresponding to the eigen value λ.

Q. Find eigen values and corresponding eigen vector of the Matrix A

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

The characteristic eqn of A is

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix}$$

$$\lambda^3 - \lambda^2(10) + \lambda(10+15+6) - 30 = 0$$

$$\lambda^3 - 10\lambda^2 + 31\lambda - 30 = 0 \qquad 8 - 40 + 62 - 30$$

$$\lambda^2(\lambda-2) - 8\lambda(\lambda-2) + 15(\lambda-2) = 0$$

$$(\lambda-2)(\lambda^2 - 8\lambda + 15) = 0$$

$\lambda - 2 = 0$ $\quad \lambda = 2,$ $\quad \lambda^2 - 5\lambda - 3\lambda + 15 = 0$

$$\lambda(\lambda-5) - 3(\lambda-5) = 0$$

$$(\lambda-3)(\lambda-5) = 0$$

$$\lambda = 2, 3, 5$$

Hence their exist three linearly indep. eigen vectors

Let $x_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ be the E.V of A

Corresponding to $\lambda = 2$

$$AX_1 = \lambda X_1$$
$$[A - 2I]X_1 = 0$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = 0$$

$R_3 \to R_3 - \frac{R_2}{2}$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = 0$$

$r = 2 \quad n = 3$

$n - r = 1$ variable is arbitrary Constant

$$x_1 + y_1 + 4z_1 = 0$$
$$6z_1 = 0$$
$$z_1 = 0$$

Let $y_1 = c$ $\qquad x_1 + c = 0$

$x_1 = -c$.

Hence

$$X_1 \begin{bmatrix} -c \\ c \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Let $X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ be the E.V of A corresponding to $\lambda = 3$

$$AX_2 = \lambda X_2$$

$$[A - 3I]X_2 = 0$$

$$\begin{bmatrix} 0 & 14 & \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = 0$$

$R_2 \to R_2 + R_1$

$$\begin{bmatrix} 0 & 14 & \\ 0 & 0 & 10 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = 0$$

$R_3 \to R_3 - \frac{R_2}{5}$

$$\begin{bmatrix} 0 & 14 & \\ 0 & 0 & 10 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = 0$$

$r = 2, \quad n = 3$

$n - r = 1$ variable is A.C.

$y_2 + z_2 = 0$

$10 z_2 = 0$

$z_2 = 0$

$z_2 = 0, \quad y_2 = 0$

$x_2 = c$

Hence eigen vector is $X_2 = \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

let $X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$ be the E.V of A Corrp.

to $\lambda = 5$

$$AX_3 = \lambda X_3$$

$$[A - 5I]\, X_3 = 0$$

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = 0$$

$r = 2, \quad n = 3$

$n - r = 1$ variable is A.C

$$-2x_3 + y_3 + 4z_3 = 0$$

$$-3y_3 + 6z_3 = 0$$

let $z_3 = c$ $\qquad y_3 = 2z_3 = 2c_1$

$x_3 = 3c_1$

Hence $X_3 = \begin{bmatrix} 3c_1 \\ 2c_1 \\ c_1 \end{bmatrix}$ or $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

Quick Method

2+2+3

Q. $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

2(2) − 1(1)
2 · 4 − 1

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix}$$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0 \qquad 1-5+7-3=0$$

$$\lambda^2(\lambda-1) - 4\lambda(\lambda-1) + 3(\lambda-1) = 0$$

$$\lambda = 1, 1, 3$$

Let $X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ be the E.V of A corresp. to the $\lambda = 1$

$$AX_1 = \lambda X_1$$

$$(A - \lambda I)X_1 = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = 0$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = 0$$

$r = 1, \quad n = 3$

$n - r = 2$ variables must have arbitrary const value

2 linear independent E.V.

$x_1 + y_1 + z_1 = 0$

Let $z_1 = c_1$ $y_1 = c_2$, $x = -c_1 - c_2$

Hence

$$X_1 = \begin{bmatrix} -c_1 - c_2 \\ c_2 \\ c_1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \& \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

⇒(*) Cayley Hamilton Theory :

Every Sq. matrix satisfied its own characteristic eq^n.

Let A be any Sq. matrix of order n and its characteristic Polynomial is -

$$|A - \lambda I| = (-1)^n \{\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots a_n\} = 0$$

To Prove

$$(-1)^n A^n + a_1 A^{n-1} + a_2 A^{n-2} \dots a_n I = 0$$

Let P be the Adjoint of Matrix $[A - \lambda I]$

$$P = P_1\lambda^{n-1} + P_2\lambda^{n-2} + \dots P_n$$

where $P_1, P_2, P_3, \dots, P_n$ are all square matrices of order n after equating the coefficient of differents powers of λ.

$$\frac{1}{A} = \frac{\text{Adj}(A)}{|A|}$$

$$\frac{I}{[A-\lambda I]} = \frac{P}{|A-\lambda I|}$$

$$|A-\lambda I|\, I = [P_1\lambda^{n-1} + P_2\lambda^{n-2} + \cdots P_n I][A-\lambda I]$$

$$[(-1)^n\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots a_n] I = (P_1\lambda^{n-1} + P_2\lambda^{n-2} + \cdots P_n I][A-\lambda I]$$

equating the coefficient of different power of λ.

$$(-1)^n I = -P_1 I$$
$$a_1 I = AP_1 - P_2 I$$
$$a_2 I = AP_2 - P_3 I$$
$$\vdots$$
$$a_{n-1} I = AP_{n-1} - P_n I$$
$$a_n I = AP_n I$$

multiplying by $A^n, A^{n-1}, A^{n-2} \cdots I$ and adding

$$(-1)^n A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_n I = 0$$

Ta - 1

Q.1 Find the Rank of mat by Reducing it Normal Form

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$$A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & -1 & 3 & 2 \\ 3 & -5 & 2 & 2 \\ 6 & -3 & 8 & 6 \end{bmatrix}$$

(2) Determine the values of a and b for which the system

$$\begin{bmatrix} 3 & -2 & 1 \\ 5 & -8 & 9 \\ 2 & 1 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ 3 \\ -1 \end{bmatrix}$$

has (i) unique sol^n

(ii) no sol^b

(iii) infinite many sol^n

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(3) Find the inverse of A.

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Pag - 34 Eb-12.

By elementary Transformation

(4) Find the eigen values of

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$$A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$$

Sol^n:

$$A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & -1 & 3 & 2 \\ 3 & -5 & 2 & 2 \\ 6 & -3 & 8 & 6 \end{bmatrix}$$

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$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$
$R_4 \to R_4 - 6R_1$

$$\begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & -7 & -5 & -2 \\ 0 & -20 & -10 & -4 \\ 0 & -21 & -40 & -6 \end{bmatrix}$$

$C_2 \to C_2 - 3C_1$
$C_3 \to C_3 - 4C_1$
$C_4 \to C_4 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & -5 & -2 \\ 0 & -20 & -10 & -4 \\ 0 & -21 & -40 & -6 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & -5 & -2 \\ 0 & -20 & -10 & -4 \\ 0 & -1 & -30 & -2 \end{bmatrix}$$

$-5 - 180$

$R_2 \to R_2 - 6R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & & \\ & & & \\ & & & \end{bmatrix}$$

$R_4 \to R_4 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & -5 & -2 \\ 0 & -20 & -10 & -4 \\ 0 & 6 & -25 & 0 \end{bmatrix}$$

~~$R_2 \to$~~

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & -5 & -2 \\ 0 & -6 & 0 & 0 \\ 0 & 6 & -25 & 0 \end{bmatrix}$$

$R_4 \to R_4 + R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & -5 & -2 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -25 & 0 \end{bmatrix}$$

$R_2 \to R_2 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -5 & -2 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -25 & 0 \end{bmatrix}$$

$R_2 \to R_2 \times -1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 2 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -25 & 0 \end{bmatrix}$$

~~$R_3 \to R_3 + 6R_2$~~

$R_3 \to \frac{R_4}{-25}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 2 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & -6 & 0 & 0 \end{bmatrix}$$

$R_6 \to R_6 / -6$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

~~$R_4 \to R_4$~~

$R_2 \to R_2 - 5R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \to R_4 \to R_4 - R_2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{pmatrix} I_3 & 0 \\ 0 & 0 \end{pmatrix}$$

Rank = 3

(2)

$$\begin{bmatrix} 3 & -2 & 1 \\ 5 & -8 & 9 \\ 2 & 1 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ 3 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & b \\ 5 & -8 & 9 & 3 \\ 2 & 1 & a & 1 \end{array}\right]$$

$9-(5-5a)$
$9-5+5a$
$3-(5b-1)$
$3-5b+5$
$8-5b$
$a-(2-2a)$
$a-2+2a$
$3a-2$
$1-(2b-2)$
$1-2b+2$
$3-2b$

$R_1 \to R_1 - R_3$

$$\begin{bmatrix} 1 & -3 & 1-a & b-1 \\ 5 & -8 & 9 & 3 \\ 2 & 1 & a & 1 \end{bmatrix}$$

$R_2 \to R_2 - 5R_1$
$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & -3 & 1-a & b-1 \\ 0 & 7 & 4+5a & 8-5b \\ 0 & -5 & 3a-2 & 3-2b \end{bmatrix}$$

$R_2 \to R_2/7$

$$\begin{bmatrix} 1 & -3 & 1-a & b-1 \\ 0 & 1 & 4+5a/7 & 8-5b/7 \\ 0 & -5 & 3a-2 & 3-2b \end{bmatrix}$$

$R_3 \to R_3 + 5R_2$

$$\begin{bmatrix} 1 & -3 & 1-a & b-1 \\ 0 & 1 & \frac{4+5a}{7} & 8-5b/7 \\ 0 & 0 & \frac{26a-10}{7} & \frac{61-39b}{7} \end{bmatrix}$$

$21a-14 \neq 4+5a$

$= \frac{26a-10}{7}$

$\frac{46a+6}{7}$

(ii) unique Sol^n:

$$\frac{46a+6}{7} \neq 0 \qquad \frac{61-39b}{7} \neq 0$$

$a \neq -3$

(ii) no Sol^n $\Rightarrow$

$$\frac{46a+6}{7} = 0 \qquad \frac{61-39b}{7} \neq 0$$

$a = -3$, $b \neq \frac{1}{3}$ $\quad$ $a = \frac{-6}{46}$ $\quad$ $a = -3$ $\quad$ $b \neq \frac{1}{3}$

(iii) ⓐ $\frac{46a+6}{7} = 0 \qquad \frac{61-39b}{7} = 0$

$a = -3$, $b = \frac{1}{3}$ $\quad$ $a = \frac{-6}{46}$ $\quad$ $b = \frac{61}{39}$

③
$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$R_4 \leftrightarrow R_1$
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} A$$

$R_2 \to R_2 - 1R_1$
$R_3 \to R_3 - 2R_1$
$R_4 \to R_4 - R_1$
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 1 & 0 & 0 & -1 \end{bmatrix} \quad -2+4$$

$R_2 \to R_2 - R_4$
$R_3 \to R_3 - 2R_4$
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 2 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

$R_1 \to R_1 - R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 2 \\ 1 & 0 & 0 & -1 \end{bmatrix} A$$

$R_2 \to R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 2 \\ 1 & 0 & 0 & -1 \\ -2 & 0 & 1 & 2 \\ -1 & 1 & 0 & 0 \end{bmatrix} A$$

$R_4 \to R_4 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 2 \\ 1 & 0 & 0 & -1 \\ -2 & 0 & 1 & 2 \\ -2 & 1 & 0 & 1 \end{bmatrix} A$$

$R_3 \to R_3 - R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 2 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 1 \\ -2 & 1 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 + R_3$

$R_3 \to R_3/-1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 2 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ -2 & 1 & 0 & 0 \end{bmatrix} A.$$

$$A^{-1} = \begin{bmatrix} -1 & 0 & 0 & 2 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ -2 & 1 & 0 & 0 \end{bmatrix}$$

(4) $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

$-21+20$

$\vec{21}-18$

$-9+20$

$$|A-\lambda I| = \begin{vmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{vmatrix}$$

$$\lambda^3 - \lambda^2(7) + (-1+6+11)\lambda -$$

$3(-21+20)-10(-14+12)-5(-10+9)$

$3(-1) - 10(-2) - 5(-1)$

$-3 + 20 + 5$

$20 \quad 20$

$3(-21+20)-10(-14+12)+5(-10+9)$

$3(-1)-10(-2)+5(-1)$

$-3-5+20$

12

Mod. Imp.

Q. Verify Cayley Hamilton theorem for the matrix A.

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

Hence find A^{-1}.

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & -2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix}$$

$$\lambda^3 - \lambda^2(1) + \lambda(-4+1+(-1)) - (-4) = 0$$

$$\lambda^3 - \lambda^2 - 4\lambda + 4 = 0$$

∴ By C.H.T

$$A^3 - A^2 - 4A + 4I = 0$$

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} - \begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^3 - A^2 - 4A + 4I = 0$$

$$A^2 - A - 4I + 4A^{-1} = 0$$

$$A^{-1} = \frac{1}{4}\left[-A^2 + A + 4I\right]$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 4 & 4 & -4 \\ -2 & -1 & 3 \\ -2 & 1 & 1 \end{bmatrix}$$ //

* find the characteristic eq^n of the matrix A.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

4+1=3

verify Cayley Hamilton theorem

Hence Proof that

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

$$\lambda^3 - (5)\lambda^2 + (2+3+2)\lambda - 3 = 0$$

by C.H.T

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

$$A^3 - 5A^2 + 7A - 3I = 0$$

$$A^5\left[A^3 - 5A^2 + 7A - 3I\right] + A\left[A^3 - 5A^2 + 7A - 3I\right] + A^2 + A + I$$

$$= A^2 + A + I =$$

Q. Find the non-singular matrices P, Q such that PAQ is a normal form

$$A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} \qquad A = PAQ$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3 \begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix}$$

$\lambda = 1, 1, 3$ → not model Matrix

Q. Find the e.value

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

$$\lambda^3 - 7\lambda^2 + (4+3+4)\lambda - (+5) = 0$$

$$\lambda^3 - 7\lambda^2 + 11\lambda + 3 = 0$$

~~$\lambda^3 - 7\lambda^2 + 11\lambda + 5 = 0$~~

$$\lambda^2(\lambda - 1) - 6\lambda(\lambda - 1) + 5(\lambda - 1) = 0$$

$$(\lambda - 1)(\lambda^2 - 6\lambda + 5)$$

$$\lambda = 1, 1, 5$$

let $X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ be the E.V of A Corr. to $\lambda = 1$

$[A - I]X_1 = 0$ $\quad AX_1 - \lambda X_1 = 0$

$(A - \lambda I)X_1 = 0$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = 0$$

$3 - 1 = 2$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = 0$$

$n = 3$

$r = 1$

$3 - 1 = 2$ var. ars. const

$x_1 + 2y_1 + z_1 = 0$

$x_1 + 2y_1 + z_1 = 0$

let $z_1 = c_1$, $y_1 = c_2$

$x_1 = -2c_2 - c_1$

$$X_1 = \begin{bmatrix} -2c_2 - c_1 \\ c_2 \\ c_1 \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

* Proof that the E.V of ~~Hermi~~ Hermitian matrix are always ~~or~~ real.

let A is mat.

$$A = A^{\theta}$$

To Prove $\boxed{\lambda = \bar{\lambda}}$

if X is E.V of A Corr. E.V of λ

$$AX = \lambda X$$

$$X^{\theta} A X = X^{\theta} \lambda X \quad \text{---} (1)$$

Transpose conjugate

$$(AX^{\theta}X)^{\theta} = (\lambda X X^{\theta})^{\theta}$$

$$[X^\theta A X]^\theta = [X^\theta A X]^\theta$$

$$X^\theta A^\theta (X^\theta)^\theta = X^\theta \lambda^\theta (X^\theta)^\theta$$

$$X^\theta A^\theta X = X^\theta \lambda^\theta X \quad - (1)$$

$$X^\theta A X = \bar{\lambda} X^\theta X \quad - (2)$$

from (1) and (2)

$$X^\theta \lambda X = \bar{\lambda} X^\theta X$$

$$X^\theta X (\lambda - \bar{\lambda}) = 0$$

Always non zero

$$\lambda - \bar{\lambda} = 0$$

$$\boxed{\lambda = \bar{\lambda}}$$

* Prove that E.V of skew-Hermition matrix are eigther purely imaginary or zero.

If A is skew Hermition matrix.

(iA)

$$A^\theta = -A \text{ (Given)}.$$

$$(iA)^\theta = A^\theta i^{*\theta}$$

$$= -A(-i)$$

$$= iA$$

iA is H.M

→ Let A is a Skew Hermition matrix and λ is any E.V of A.

iA = Hermition matrix

and $i\lambda$ is E.V of iA

Since we know that E.V of H.M is always real therefore $i\lambda$ must be a Real no. and this is Possible only when either λ is purely imaginery or zero.

* Similar Matrices :

if A and B are two matrices of order n. then B is said to be similar to matrix A if there exist a non-singular matrix P. such that

$$\boxed{B = P^{-1}AP}$$

* Diagonalization :

Diagonalization is a process to reduce a matrix A into D where D is Diagonal matrix.

if $D = P^{-1}AP$

then we can say A and D are similar matrix. where P is called Modal matrix.

Q. To A if $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ find the modal matrix and the resultant diagonal matrix of A.

Chara. eq^n of A

$$\lambda^3 - 3\lambda^2 + (1+(-2)+0)\lambda - (-3) = 0$$
$$\lambda^3 - 3\lambda^2 - \lambda + 3 = 0$$

$$\lambda^2(\lambda-1) - 2\lambda(\lambda-1) - 3(\lambda-1) = 0$$
$$(\lambda-1)(\lambda^2 - 2\lambda - 3) = 0$$

$(\lambda^2 - 3\lambda + \lambda - 3)(\lambda + 1) = 0$

$\lambda = 3, -1, 1$

$\lambda = 1 \qquad X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$

$(A - I)X_1 = 0$

$$\begin{bmatrix} 0 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = 0$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = 0$$

$n = 3, \; r = 2$

$3 - 2 = 1$ ∞ is arb. const

$x_1 + y_1 + z_1 = 0$

$2y_1 - 2z_1 = 0$ let $z_1 = c$

so, $y_1 = c$

$x_1 = -2c$

$$X_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

for $\lambda = -1$ $\quad X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$

$$[A+I]X_2 = 0$$

$$\begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & 1 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = 0$$

$R_2 \to R_1 - \frac{R_1}{2}$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = 0$$

$n - r = 1$ var. is arb. const.

$$x_2 + y_2 - z_2 = 0$$
$$2y_2 + 2z_2 = 0$$

$z_2 = c_1 \quad y_2 = -c_1$

$x_2 = +2c_1$

$$X_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

for $\lambda = 3$ $\quad X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$

$$(A - 3I)X_3 = 0$$

$$\begin{bmatrix} -2 & 2 & -2 \\ 1 & -1 & 1 \\ -1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & +1 & -1 \\ -2 & 1 & -1 & 1 \\ -1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} +1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = 0$$

$n - r = 1$ is arb. Const.

$$x_3 - y_3 + z_3 = 0$$
$$-2y_3 - 2z_3 = 0$$

$z_3 = c_3$

$y_3 = -c_3$

$x_3 = -2c_3$

$$X_3 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -2 & 2 & -2 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 0 & 1/2 \\ -1/4 & -1/2 & 0 \end{bmatrix}$$

(vi) Find the modal matrix of $A =$ which diagonalises A also find D.

* Orthogonal Transformation :-

If A be any symmetric, then $A = A'$ and orthogonal

then

$$AA' = I \quad \text{---(1)}$$

$$AA^{-1} = I \quad \text{---(2)}$$

$$\boxed{A' = A^{-1}}$$

Also we know that,

$$P'AP = D$$

where P is modal matrix

If we normalise each eigen vector and use them to form a normalise modal matrix N then

$$N'AN = D$$

The transforming A into D by means of

$$N'AN = D$$

is called orthogonal transformation

- To Normalsis eigen vector divide each element of the vector by the +ve square root of Sum of the square of all the element of eigen vector

Que:- Find All the eigen values and eigen vectors of the matrix A

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Also Diagonalis the matrix A By means of orthogonal transformation.

Sol^{n}:-

$$\lambda^3 - 6\lambda^2 + (3+3+3)\lambda - 4 = 0$$

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

$$\lambda^2(\lambda-1) - 5\lambda(\lambda-1) + 4(\lambda-1) = 0$$

$$(\lambda-1)(\lambda^2 - 5\lambda + 4) = 0$$

$$\lambda = 1, 1, 4 \quad \text{(eigen values of A)}$$

Let $X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ be the eigen v of A Corr. to $\lambda = 1$

$$AX_1 = \lambda X_1$$

$$(A - \lambda I)X_1 = 0$$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = 0$$

$R_3 \to R_3 - R_1$
$R_2 \to R_2 + R_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = 0$$

$r=1$, $n=3$ $\quad n-r=2$ arb. are const.

$$x_1 - y_1 + z_1 = 0$$

$z_1 = c_1 \quad y_1 = c_2$

$$x_1 = c_2 - c_1$$

$$X_1 = \begin{bmatrix} c_2 - c_1 \\ c_2 \\ c_1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Let $X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ is e.v corresponding to $\lambda = 4$

$$[A - 4I] X_2 = 0$$

$$\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = 0$$

$R_2 \to R_2 - R_1/2$
$R_3 \to R_3 + R_1/2$

$$\begin{bmatrix} -2 & -1 & 1 \\ 0 & -3/2 & -3/2 \\ 0 & -3/2 & -3/2 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = 0$$

$$R_3 \to R_3 - R_2 \quad \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3/2 & -3/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = 0$$

$r = 2, \; n = 3 \qquad n - r = 1$ Vol. arb. is constant

$$-2x_2 - y_2 + z_2 = 0$$
$$y_2 + z_2 = 0$$

let $z_2 = c$

$$y_2 = -c$$

$$x_2 = \frac{-c - c}{-2} = c$$

$$x_2 = c$$

$$X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

modal matrix $(P) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$

Find out N.

$$N = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \; 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & -1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{3} \end{bmatrix}$$

$$N' = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$N'AN = [\quad][\quad][\quad]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$