13/11/12 * Definite & Improper Integrals *

Ques $\int_0^1 \frac{1-x^2}{1+x^2}\,dx$

$1+x^2 \overline{)1-x^2(-1}$

$\frac{-1-x^2}{2}$

$= -1 + \frac{2}{1+x^2}$

$= \int_0^1 \left(-1 + \frac{2}{1+x^2}\right) dx$

$= \left[-x + 2\tan^{-1}x\right]_0^1$

$= -1 + 2\tan^{-1}1$

$= -1 + \frac{2\pi}{4}$

(2) $I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}$ — (1)

$I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right) dx}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}$

$= \int_0^{\pi/2} \frac{\cos x\,dx}{\sin x + \cos x}$ — (2)

add (1) and (2)

$$2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x}\, dx$$

$$2I = \pi/2$$

$$I = \pi/4 \quad \underline{\text{Ans}}$$

(3) $$I = \int_0^{\pi/2} \log(\sin x)\, dx \qquad \text{---} (1)$$

$$I = \int_0^{\pi/2} \log(\sin(\pi/2 - x))\, dx$$

$$I = \int_0^{\pi/2} \log \cos x\, dx \qquad \text{---} (2)$$

add (1) and (2)

$$2I = \int_0^{\pi/2} \log \sin x + \log \cos x\, dx$$

$$2I = \int_0^{\pi/2} \log \sin x \cos x\, dx$$

$$2I = \int_0^{\pi/2} (\log \sin 2x - \log 2)$$

$$= \int_0^{\pi/2} \log \sin 2x - \log \int_0^{\pi/2} dx$$

$$\int_0^{\pi/2} \log \sin 2x \, dx \qquad \frac{1}{2}\int \log \sin t \, dt$$

$\log \sin t = t$

$\frac{1}{\sin t}$

Let $2x = t$

$\Rightarrow 2dx = dt$

$$\int_0^{\pi} \frac{\log \sin t \, dt}{2}$$

$$2\int_0^{\pi/2} \frac{\log \sin t}{2} \, dt$$

$$2I = I - \log 2 (\pi/2)$$

$$I = 0 - \pi \frac{\log 2}{2}$$

$x^2 = t$

$2x dx = dt$

$\therefore$

$1 + x^2 = t$

$2x dx = dt$

Q. $\int_0^1 \frac{\log(1+x)}{1+x^2}$

$x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

$$\int_0^{\pi/4} \frac{\log(1+\tan\theta)}{\sec^2\theta} \sec^2\theta \, d\theta$$

$$\int_0^{\pi/4} \log(1+\tan\theta) \, d\theta$$

$$= \int_0^{\pi/4} \log\left(1 + \tan(\pi/4 - \theta)\right) d\theta$$

$$= \int_0^{\pi/4} \log\left(1 + \left(\frac{1-\tan\theta}{1+\tan\theta}\right)\right) d\theta$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right) d\theta$$

$$I = \int_0^{\pi/4} \log 2 - \int_0^{\pi/4} (\log(1+\tan\theta)) d\theta$$

$$2I = \int_0^{\pi/4} \log 2 \, d\theta$$

$$2I = \frac{\pi}{4} \log 2$$

$$I = \frac{\pi}{8} \log 2$$

* Improper Integral :
the definate integral :

$\int_a^b f(x)\,dx$ is called improper integral

(if either a or b or both are infinite so, that the integration is bounded interval of

but f is bounded ,

then $\int_a^b f(x)\,dx$ is improper integral

of first kind.

fnot ∞ for (1, ∞) value

Ex: $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ — Limit is unbounded, fn is bounded.

⇒ if the definate integral of a

$\int_a^b f(x)\,dx$ is such that the interval of integration is finite but f has one or more points of infinite discontinuity or f is unbounded then

$\int_a^b f(x)\,dx$ is called improper integral of 2nd Kind

for Ex: $\int_0^1 \frac{dx}{x^2}$ or $\int_1^2 \frac{dx}{(x-2)}$ — Limit is finite fn becom unbounded

⇒ if the definate integral

$\int_a^b f(x)\,dx$ is such that the interval of integration is unbounded and f is also unbounded the the integral $\int_a^b f(x)\,dx$ is called improper integral of 3rd kind.

Ex: $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$

1st Kind Solve:

- Examine the convergence of the improper integrals

$$\int_1^{\infty} \frac{dx}{\sqrt{x}}$$

$$\lim_{t\to\infty} \int_1^{t} \frac{dx}{\sqrt{x}} = \lim_{t\to\infty} \left[2\sqrt{x}\right]_1^{t}$$

$$\lim_{t\to\infty} \left[2\sqrt{t} - 2\right] = \infty$$ divergent

infinite $-\infty$] divergent

finite → convergent

Que $$\int_0^{\infty} \sin x \, dx$$

$$\lim_{t\to\infty} \int_0^{t} \sin x \, dx$$

$$\lim_{t\to\infty} \left[-\cos x\right]_0^{t}$$

$$\lim_{t\to\infty} \left[-\cos t + 1\right]$$

Integral is oscillatory

$\frac{\infty}{\infty}$ = undefined

Que: $\int_2^{\infty} \frac{2x^2}{x^4-1}\,dx$

$\lim_{t\to\infty} \int_2^{t} \left(\frac{2x^2}{x^4-1}\right) dx$

$\int \frac{2x^2}{(x^2-1)(x^2+1)}$ $= \frac{(x^2+1)+(x^2-1)}{(x^2-1)(x^2+1)}$

$= \frac{1}{x^2-1} + \frac{1}{x^2+1}$

$= \lim_{x\to\infty} \int_2^{t} \left(\frac{1}{x^2-1} + \frac{1}{x^2+1}\right) dx$

$\lim_{x\to\infty} \left(\int_2^{t} \frac{1}{x^2-1} + \int_2^{t} \frac{1}{x^2+1} \right)$

$\frac{1}{(x-1)(x+1)}$

$(x-1)-(x+1)$

$\lim_{t\to\infty} \left(\frac{1}{(x-1)(x+1)}\,dx + (\tan^{-1}x)_2^t \right)$

$= \lim_{t\to\infty} \left[\frac{(x+1)-(x-1)}{(x-1)(x+1)}\,dx + \tan^{-1}t - \tan^{-1}2 \right)$

$\lim_{t\to\infty} \left(\frac{1}{2}\log\left(\frac{x-1}{x+1}\right)\right)_2^t + \tan^{-1}t - \tan^{-1}2$

$\lim_{t\to\infty} \left[\log\left(\frac{1-\frac{1}{t}}{1+\frac{1}{t}}\right) - \log\left(\frac{1}{3}\right) + \tan^{-1}t - \tan^{-1}2 \right]$

$\log 0 = 1$

$\frac{1}{2}(\log 1 - \frac{1}{2}\log 1 - \frac{1}{2}\log 3 - \frac{\pi}{2} - \tan^{-1} 2$

= Finite value

= Convergent

Que: $\int_0^1 \frac{dx}{\sqrt{x}}$.

$\lim_{\varepsilon \to 0^+} \int_{0+\varepsilon}^{1} \frac{dx}{\sqrt{x}}$

$\lim_{\varepsilon \to 0^+} [2\sqrt{x}]_{0+\varepsilon}^{1}$

$\lim_{\varepsilon \to 0^+} (2 - 2\sqrt{\varepsilon})$ = Finite value

= Convergent

Que: $\int_0^{1/E} \frac{dx}{x(\log x)^2}$

$\log x = t$

$\frac{1}{x} dx = dt$

$\lim_{\varepsilon \to 0^+} \int_{0+\varepsilon}^{1/E} \frac{1}{t^2} dt$

t^{-2}

$\frac{t^{-1}}{-2}$

$= -\frac{1}{2t}$

$\lim_{\varepsilon \to 0^+} \left[\frac{-1}{2t}\right]_{\log \varepsilon}^{1} = -\log \varepsilon$

$$\lim_{\epsilon \to 0^+} \left[\frac{-1}{at} \right]_{-\log \epsilon}^{1} \Big/ \left[\frac{\log x}{-1} \right]_{0+\epsilon}^{1}$$

$$\lim_{\epsilon \to 0^+} \left[\frac{-1}{\log(\frac{1}{\epsilon})} + \frac{1}{\log(\epsilon)} \right]$$

$$= \text{finite}$$

$$= \text{Convergent.}$$

Que.

$$\int_0^2 \frac{dx}{\sqrt{4-x^2}}$$

$$\lim_{\epsilon \to 0^+} \int_{0+\epsilon}^{2-\epsilon} \frac{dx}{\sqrt{4-x^2}}$$

confirm

$$\lim_{\epsilon \to 0^+} \left[\frac{1}{2} \sin^{-1}\left(\frac{x}{2}\right) \right]_0^{2-\epsilon}$$

$$\lim_{\epsilon \to 0^+} \left[\frac{1}{2} \sin^{-1}\left(\frac{2-\epsilon}{2}\right) \right] = \frac{1}{2} \sin^{-1} 1$$

$$= \pi/4$$

$$\text{finite}$$

$$= \text{Convergent}$$

3rd lecture

Que: $\int_{-\infty}^{\infty} e^{-x}\,dx$

$$= \int_{-\infty}^{0} e^{-x}\,dx + \int_{0}^{\infty} e^{-x}\,dx$$

$$\lim_{t_1 \to -\infty} \int_{t_1}^{0} e^{-x}\,dx + \lim_{t_2 \to \infty} \int_{0}^{t_2} e^{-x}\,dx$$

$$\lim_{t_1 \to -\infty} \left[-e^{-x}\right]_{t_1}^{0} + \lim_{t_2 \to \infty} \left[-e^{-x}\right]_{0}^{t_2}$$

$$\lim_{t_1 \to -\infty} \left[-1 + e^{-t_1}\right] + \lim_{t_2 \to \infty} \left[-e^{-t_2} + 1\right]$$

$$= -1 + \infty + 0 + 1$$

$$= \infty$$

Integrals is divergent

Que: $\int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2}$ First trial

Que: $\int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}} = \int_{-\infty}^{0} \frac{dx}{e^x + e^{-x}} + \int_{0}^{\infty} \frac{dx}{e^x + e^{-x}}$

$\frac{e^x}{(e^{2x}+1)}$

$$\lim_{t_1 \to -\infty} \int_{t_1}^{0} \frac{dx}{e^x + e^{-x}} + \lim_{t_2 \to \infty} \int_{0}^{t_2} \frac{dx}{e^x + e^{-x}}$$

$$= \frac{e^x}{e^{2x}+1}$$

$$e^x = t$$
$$e^x dx = dt$$

$$\lim_{t_1 \to -\infty} \int_{t_1}^{0} \frac{dt}{t^2+1} + \lim_{t_2 \to \infty} \int_{0}^{t} \frac{dt}{t^2+1}$$

$$\lim_{t_1 \to -\infty} \left[\tan^{-1} e^x\right]_{t_1}^{0} + \lim_{t_2 \to \infty} \left[\tan^{-1} e^x\right]_{0}^{t_2}$$

$$\lim_{t_1 \to -\infty} \left[\tan^{-1} 1 - \tan^{-1} e^{t_1}\right] + \lim_{t_2 \to \infty} \left[\tan^{-1} e^{t_2} - \tan^{-1} 1\right]$$

$$= \left[\frac{\pi}{4} - 0 + \frac{\pi}{2} - \frac{\pi}{4}\right]$$

$$= \pi/2 \quad \text{Ans}$$

* Beta & Gamma func^n :

Beta f^n: The definate integral $\int_0^1 x^{m+} (1-x)^{n+} dx$

is called Beta f^n , $m>0, n>0$

and it is denoted by $B(m,n)$

$$B(m,n) = \int_0^1 x^{m+} (1-x)^{n+} dx.$$

1. $B(m,n) = B(n,m)$

$$B(m,n) = \int_0^1 x^{m+} (1-x)^{n+} dx$$

$$= \int_0^1 (1-x)^{m+} (1-(1-x))^{n+} dx$$

$$= \int_0^1 x^{n+} (1-x)^{m+} dx$$

$$= B(n,m).$$

Hence Beta f^n is symmetrical.

Q. Express the following integrals in terms of Beta f^n.

11. $\int_0^1 \frac{x^2}{\sqrt{1-x^5}} = \int_0^1 x^2 (1-x^5)^{-1/2}\, dx$

Let $x^5 = y \Rightarrow 5x^4\, dx = dy$

$\int_0^1 x^4 x^{-2} (1-x^5)^{-1/2}\, dx$

$a - x = t$
$\Rightarrow -dx = dt$

$x = t$
$\Rightarrow dx = dt$

$\frac{1}{5}\int_0^1 y^{-2/5} (1-y)^{-1/2}\, dy$

$\frac{1}{5}\int_0^1 y^{3/5-1} (1-y)^{1/2-1}$

Qu. $\int_0^a (a-x)^{m-1} x^{n-1}\, dx = a^{m+n-1} B(m,n)$

Let $x = ay$

$(a-x)$
$(1-y)$

$\int_0^1 a^{m-1} (1-y)^{m-1} (ay)^{n-1} a\, dy$

$a^{m+n-1} \int_0^1 y^{n-1} (1-y)^{m-1}\, dy$

$a^{m+n-1} \beta(n,m)$

$a^{m+n-1} \beta(m,n)$

Que $\int_a^b (x-a)^{m+1} (b-x)^{n+1} dx = (b-a)^{m+n+1} B(m,n)$

~~$x = a + y$~~

$x = a + (b-a)y$

$x = a, \; y = 0$
$y = b, \; y = 1$

~~$x = (a + (b)\, y$~~

$dx = (b-a)\, dy$

$\int_0^1 (b-a)^{m+1} y^{m+1} (b-a-(b-a)y)^{n+1} (b-a)\, dy$

$(b-a)^{m+n+1} \int_0^1 y^{m+1} (1-y)^{n+1} dy$

$(b-a)^{m+n+1} B(m,n)$

Que: $\int_0^1 x^m (1-x^2)^n dx$

$x^2 = y$
$2x\, dx = dy$

~~ans~~ $= \frac{1}{2} B\left(\frac{m+1}{2}, n+1\right)$

Imp Que: $= \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$

$= \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

$p > -1, \; q > -1$

$\sin\theta = y$

Deduce $\int_0^2 x^4 (8-x^3)^{1/3} dx = \frac{16}{3} B\left(\frac{5}{3}, \frac{2}{3}\right)$

$\sin^2\theta = \frac{y}{8}x$

$2\sin\theta\cos\theta\, d\theta = dx$

$= \int_0^{\pi/2} \sin^{p-1}\theta \cos^{q-1}\theta \, d\theta \; \sin\theta\cos\theta\, d\theta$

$= \frac{1}{2}\int_0^1 x^{\frac{p-1}{2}} (1-x)^{\frac{q-1}{2}} dx$

$= \frac{1}{2}\int_0^1 x^{\frac{p+1}{2}-1} (1-x)^{\frac{q+1}{2}-1} dx$

$= \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) \quad p>-1, q>-1$

Deduce $x^3 = 8\sin^2\theta$

$x = 2\sin^{2/3}\theta$

$dx = \frac{4}{3}\sin^{-1/3}\theta \times \cos\theta\, d\theta$

$\int_0^{\pi/2} \frac{16}{2} \sin^{8/3}\theta (\cos^2\theta)^{-1/3} \times \frac{4}{3} \sin^{-1/3}\theta \cos\theta\, d\theta$

$\frac{32}{3}\int_0^{\pi/2} \sin^{7/3}\theta \cos^{1/3}\theta\, d\theta$

$$\frac{32}{3}\left[\frac{1}{2}\beta\left(\frac{10}{6},\frac{4}{6}\right)\right]$$

$$\frac{16}{3}\left[\beta\left(\frac{5}{3},\frac{2}{3}\right)\right]$$

Que: In that p and $q > 0$

(i) $$\frac{\beta(p,q+1)}{q} = \frac{\beta(p+1,q)}{p} = \frac{\beta(p,q)}{p+q}$$

(ii) $$\beta(p,q) = \beta(p+1,q) + \beta(p,q+1)$$

$$\frac{\beta(p,q+1)}{q} = \frac{1}{q}\int_0^1 x^{p-1}(1-x)^q\,dx$$

$$= \frac{1}{q}\left[\frac{x^p}{p}(1-x)^q + \int_0^1 \frac{qx^p}{p}(1-x)^{q-1}\,dx\right]$$

$$= \frac{1}{p}\beta(p+1,q)$$

$$\frac{\beta(p+1,q)}{p} = \frac{1}{p}\int_0^1 x^p(1-x)^{q-1}\,dx$$

$$= \frac{1}{p}\int_0^1 x^{p+1}(1-(1-x))(1-x)^{q-1}\,dx$$

$$= \frac{1}{p}\left[\int_0^1 x^{p+1}(1-x)^{q+1}\,dx - \int_0^1 x^{p+1}(1-x)^{q}\,dx\right]$$

$$= \frac{1}{p}\left[\beta(p,q) - \beta(p,q+1)\right]$$

$$\beta(p,q+1)\left[\frac{1}{q}+\frac{1}{p}\right] = \frac{1}{p}\beta(p,q)$$

$$\frac{\beta(p,q+1)}{pq}(p+q) = \frac{1}{p}\beta(p,q)$$

$$= \beta(p,q+1) = \frac{\beta(p,q)}{p+q}$$

(ii) $\frac{\beta(p,q+1)}{q} = \frac{\beta(p,q)}{p+q}$

$\frac{2\beta(p,q)}{p+1}$ $\frac{\beta(p,q+1)}{q}$ $\frac{\beta(p,q)}{p}$

$\frac{\beta(p+1,q)}{p} = \frac{\beta(p,q)}{p+q}$

$\frac{2\beta(p,q)}{p+q} = p(\beta(p,q+1)$

$\beta(p,q+1) = \frac{q\,\beta(p,q)}{p+q}$

$\beta(p+1,q) = \frac{p\beta(p,q)}{p+q}$

Qu: $\frac{\beta(m+1, n)}{\beta(m, n)} = \frac{m}{m+n}$

$$\beta(m+1, n) = \int_0^1 x^m (1-x)^{n-1} dx = \beta(n, m+n)$$

$$\beta(n, m+1) = \int_0^1 x^{n-1} (1-x)^m dx$$

$$= \left[\frac{x^n}{n} (1-x)^m + \int_0^1 m \frac{x^n}{n} (1-x)^{m-1} dx \right]$$

$$= \frac{m}{n} \int_0^1 x^n (1-x)^{m-1} dx$$

$$= \frac{m}{n} \left[\int_0^1 x^{n-1} (1-(1-x))(1-x)^{m-1} dx \right]$$

$$= \frac{m}{n} \left[\int_0^1 x^{n-1} (1-x)^{m-1} dx - \int_0^1 x^{n-1} (1-x)^m dx \right]$$

$$= \frac{m}{n} \left[\beta(n, m) - \beta(n, m+1) \right]$$

$$\beta(n, m+1) = \frac{m}{n} \beta(m, n)$$

$$\beta(m+1, n)\,(m+n) = m\,\beta(m,n)$$

2nd definition

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$x = 0 \quad y = 0$

$x = \infty \quad y = \infty$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}$$

$$x = \frac{1}{y+1}$$

$$x = \frac{1}{1+y}$$

$$dx = \frac{-1}{(1+y)^2}\,dy$$

$$= \int_0^\infty \frac{1}{(1+y)^{m-1}}\left(1-\frac{1}{1+y}\right)^{n-1} \times \frac{1}{(1+y)^2}\,dy$$

$$\int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \beta(m,n)$$

Que P.T $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}} dx$

$x = \frac{a}{b} y$ $\quad x = \frac{1}{y}$ ∴

$bx = ay$

$b\,dx = a\,dy$

$\int_0^1 \frac{(\frac{ay}{b})^{m-1}}{(a+ay)^{m+n}} \times \frac{a\,dy}{b}$

$= \frac{a^m}{a^{m+n} b^m} \int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}} dy$

$\frac{1}{a^n b^m} \beta(m,n)$

* Gamma fn : the definate integral

$\sqrt{n} = \int_0^\infty e^{-x} x^{n-1} dx$ for $n>0$

is called Gamma fn. denoted by $\sqrt{n}$

(i) Prove that $\sqrt{n+1} = n\sqrt{n}$

$$\sqrt{n+1} = \int_0^\infty e^{-x} x^n \, dx$$

$$= \int_0^\infty e^{-x} \cdot x \cdot x^{n-1} \, dx$$

$$= \int_0^\infty e^{-x} (1 - (1-x)) x^{n-1} \, dx$$

$$\sqrt{n+1} = n\sqrt{n}$$

$$\sqrt{1} = \int_0^\infty e^{-x} x^0 \, dx = [-e^{-x}]_0^\infty = 1$$

$$\sqrt{n+1} = \int_0^\infty e^{-x} x^n \, dx$$

$$= \left[-e^{-x} x^n + \int_0^\infty n e^{-x} x^{n-1} \, dx\right]$$

$$\sqrt{n+1} = n\sqrt{n}$$

(ii) $\sqrt{n+1} = n!$ n is the integral

$$\sqrt{n+1} = n\sqrt{n}$$
$$\sqrt{n} = (n-1)\sqrt{n-1}$$
$$\sqrt{n-1} = (n-2)\sqrt{n-2}$$

$\overline{n+1} = n\overline{n}$ $\quad \overline{1} = 1$

$\overline{n+1} = n!$ $\quad \overline{0} = \infty$

$\overline{-1} = \infty$

$(x-1)_0^{\infty}$ $\quad x^{\infty}$

$$\overline{n+1} = n(n-1)(n-2)\cdots\overline{1}$$
$$= n(n-1)(n-2)(n-3)\cdots 1$$
$$= n!$$

* $\overline{n+1} = n\overline{n}$

$\overline{1} = 0\overline{0}$

$\overline{0} = \frac{1}{0} = \infty$

$\boxed{\overline{0} = \infty}$

* $\overline{-n} = \infty$

$\overline{n+1} = n\overline{n}$

$\overline{0} = -1\overline{-1}$

$\frac{\infty}{-1} = -\infty\overline{-1}$

Laws

* $\overline{n} = \int_0^{\infty} e^{-x} x^{n-1}\, dx$

$x = ay$

$dx = a\,dy$.

$$\int_0^{\infty} e^{-ay} (ay)^{n-1} \cdot a\, dy$$

$$= \boxed{\frac{\Gamma n}{a^n} = \int_0^\infty e^{-ay} y^{n-1}\, dy} \quad (L)$$

* $\Gamma 1/2 = \sqrt{\pi}$

$\Gamma 3/2 = \frac{1}{2}\Gamma 1/2 = \frac{1}{2}\sqrt{\pi}$

$\Gamma 11/2 = \frac{9}{2} \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi}$

$\Gamma 5/2 = \frac{3}{2} \times \frac{1}{2} \sqrt{\pi}$

(1) * $\int_0^{\pi/2} \sin^m\theta \cos^n\theta\, d\theta = \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\Gamma\frac{m+n+2}{2}}$

* ↳ Walli's formula

* Relation b/w Beta fn and Gamma fn :- Imp

P.T $B(m,n) = \frac{\Gamma m\, \Gamma n}{\Gamma(m+n)}$

$\Gamma m = \int_0^\infty e^{-x} x^{m-1}\, dx.$

$= \int_0^\infty e^{-3x} x^{m-1}\, dx = \frac{\Gamma m}{3^m}$

multiplying both side by $z^{n+} e^{-z}$ and integrating both sides w.r.t z from zero to infinity

$$\frac{\Gamma m}{z^m} \int_0^\infty e^{-z} z^{n+} dz = \int_0^\infty e^{-3x} x^{m+} \int_0^\infty e^{z} z^{n+} dz$$

$$\Gamma m \Gamma n = \int_0^\infty e^{-z(n+1)} x^{m+} dx \int_0^\infty z^{m+n+} dz$$

$$= \int_0^\infty e^{-z(n+1)} z^{m+n+} dz \int_0^\infty x^{m+} dx$$

$$\frac{\Gamma n}{q^n} = \int_0^\infty e^{-qx} x^{n+} dx$$

$$= \frac{\Gamma m+n}{(1+x)^{m+n}} \int_0^\infty x^{m+} dx$$

$$= \Gamma m+n \int_0^\infty \frac{x^{m+}}{(1+x)^{m+n}} dx$$

$$\Gamma m \Gamma n = \Gamma m+n \; B(m,n).$$

$$\frac{\Gamma m \Gamma n}{\Gamma m+n} = B(m,n)$$

Q. $\Gamma 1/2 = \sqrt{\pi}$

$$\boxed{\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}}$$

$m = n = 1/2$

$$\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{(\Gamma 1/2)^2}{\Gamma 1}$$

$$\left(\Gamma \frac{1}{2}\right)^2 = \beta\left(\frac{1}{2}, \frac{1}{2}\right)$$

$$\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \int_0^1 x^{1/2-1} (1-x)^{\frac{1}{2}-1}$$

$$= \int_0^1 x^{-1/2} (1-x)^{-1/2} \, dx$$

$(1-x)^{-1/2} = y$

Let $x = \sin^2\theta$

$dx = 2 \sin\theta \cos\theta \, d\theta$

$$\int_0^{\pi/2} \frac{2 \sin\theta \cos\theta \, d\theta}{\sin\theta \cos\theta}$$

$$= \int_0^{\pi/2} 2 \, d\theta = \pi$$

$$\left(\Gamma \frac{1}{2}\right)^2 = \pi$$

$$\boxed{\Gamma 1/2 = \sqrt{\pi}}$$

Ques: $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

$x^2 = t$

$2x\,dx = dt$

$$\int_0^\infty e^{-t} \times \frac{dt}{2\sqrt{t}}$$

$$\frac{1}{2}\int_0^\infty e^{-t} t^{1/2}\, dt$$

$$\frac{1}{2}\int_0^\infty e^{-t} t^{1/2-1}\, dt$$

$$\frac{1}{2}\Gamma 1/2 = \frac{\sqrt{\pi}}{2}$$

Wallies Formula

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta\, d\theta = \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\Gamma\frac{m+n+2}{2}}$$

Let $\sin^2\theta = x$

$2\sin\theta\cos\theta\, d\theta = dx$

$$\int_0^{\pi/2} \sin^{\frac{m}{2}}\theta \cos^{\frac{n}{2}}\theta \sin\theta\cos\theta\, d\theta$$

$$= \int_0^1 x^{\frac{m-1}{2}} (1-x)^{\frac{n-1}{2}} \frac{dx}{2}$$

$$= \frac{1}{2} \int_0^1 x^{\left(\frac{m+1}{2}-1\right)} (1-x)^{\frac{n+1}{2}-1} dx$$

$$= \frac{1}{2} \beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

$$\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right) = \frac{1}{2}\left[\frac{\left\lceil\frac{m+1}{2}\right\rceil \left\lceil\frac{n+1}{2}\right\rceil}{\left\lceil\frac{m+n+2}{2}\right\rceil}\right]$$

* Duplication formula :-

$$\lceil m \lceil m+\tfrac{1}{2} = \frac{\lceil \pi}{2^{2m-1}} \lceil 2m$$

$$\beta(m,n) = \frac{\lceil m \lceil n}{\lceil m+n}$$

$$\beta(m,m) = \frac{(\lceil m)^2}{\lceil 2m} \quad -①$$

$$\beta(m,m) = \int_0^1 x^{m-1} (1-x)^{m-1} dx$$

Let $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\, d\theta$

$$\int_0^{\pi/2} 2\sin^{2m-1}\theta \cos^{2m+1}\theta\, d\theta$$

$$2\int_0^{\pi/2} \frac{(\sin 2\theta)^{2m+1}}{2^{2m+1}}\, d\theta$$

Let $2\theta = t$

$2\,d\theta = dt$

$$\int_0^{\pi} \frac{(\sin t)^{2m+1}}{2^{2m+1}}\, dt$$

$$= \int_0^a f(x)\,dx = 2\int_0^{a/2} f(a-x)\,dx$$

$$\frac{2}{2^{2m+1}} \int_0^{\pi/2} \sin^{2m+1} t\, dt$$

$$\frac{2}{2^{2m+1}} \int_0^{\pi/2} \sin^{2m+1} t \cos^{0} t\, dt$$

$$\frac{2}{a^{2m-1}}\left[\frac{\Gamma\frac{2m}{2}\,\Gamma 1/2}{2\Gamma\frac{2m+0+2}{2}}\right]$$

$$\beta(m,m) = 2\frac{\Gamma m\sqrt{\pi}}{2^{2m-1}\Gamma\frac{m+1}{2}} \quad -(2)$$

from (1) and (2)

$$\frac{[\Gamma(m)]^2}{(\Gamma 2m)} = \frac{\Gamma m\sqrt{\pi}}{2^{2m-1}\Gamma\frac{m+1}{2}}$$

$\frac{2-1}{n}$ $2-\frac{n}{2}$

$$\Gamma m\,\Gamma\frac{m+1}{2} = \frac{\Gamma\pi}{2^{2m-1}}\Gamma 2m \quad \text{H.P.d}$$

Que: $\int_0^1 \frac{dx}{(1-x^n)^{1/2}} = \frac{\sqrt{\pi}\,\Gamma 1/n}{n\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$

let $x^n = \sin^2\theta$ $\qquad x = \sin^{2/n}\theta$

$$nx^{n-1}dx = 2\sin\theta\cos\theta\,d\theta$$

$$\int_0^{\pi/2} \frac{2\sin\theta\cos\theta}{n\sin^{\frac{2n-2}{n}}\theta(\cos\theta)}$$

$$\frac{2}{n}\int_0^{\pi/2} \sin^{\frac{2n-2}{n}}\theta\, d\theta$$

$$\frac{2}{n}\int_0^{\pi/2} \sin^{\frac{n+2}{n}-2}\theta\, d\theta$$

$$\frac{2}{n}\int_0^{\pi/2} \sin^{\frac{2}{n}+1}\theta\, d\theta$$

Ques

$$\frac{2}{n}\left[\frac{\Gamma\frac{2}{n}\,\Gamma\frac{1}{2}}{2\Gamma\left(\frac{\frac{2}{n}+1+0+2}{2}\right)}\right]$$

$$= \frac{1}{n}\Gamma\pi\,\Gamma\frac{1}{n} = \Gamma\frac{\frac{2}{n}+1}{2}$$

Ques

$$\int_0^1 \frac{(x^{m-1}+x^{n+1})}{(1+x)^{m+n}}\,dx$$

$$\int_0^1 \frac{x^{m+1}+x^{n+1}}{(1+x)^{m+n}}\,dx$$

$$\int_0^1 \frac{x^{m+1}}{(1+x)^{m+n}} + \int_0^1 \frac{x^{n+1}}{(1+x)^{m+n}}\,dx$$

Let $x = \frac{1}{y}$

$$\int_0^1 \frac{x^{n-1}\,dx}{(1+x)^{m+n}} = \int_\infty^1 \frac{\left(\frac{1}{y}\right)^{n-1}}{\left(1+\frac{1}{y}\right)^{m+n}} \times \left(-\frac{1}{y^2}\right) dy$$

$$= \int_1^\infty \frac{y^{m+n}}{y^{n+1}(1+y)^{m+n}}\,dy$$

$$= \int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$$

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$\int_0^1 \frac{x^{m-1}\,dx}{(1+x)^{m+n}} + \int_1^\infty \frac{x^{m-1}\,dx}{(1+x)^{m+n}}$$

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}$$

$$\beta(m,n)$$

$$= \frac{\Gamma m\ \Gamma n}{\Gamma m+n}$$

Que: $\int_0^{\pi/2} (\tan x)^n\,dx = \frac{\pi}{2}\sec\frac{n\pi}{2}$

Hint $\Gamma n\ \Gamma(1-n) = \frac{\pi}{\sin n\pi}$

$$\int_0^{\pi/2} \sin^n x \cos^{-n} x \, dx$$

$$\frac{\Gamma\frac{n+1}{2}\,\Gamma\frac{-n+1}{2}}{2\Gamma\frac{n-n+2}{2}} = \frac{\Gamma\frac{n+1}{2}\,\Gamma\frac{1-n}{2}}{2}$$

$$= \frac{\Gamma\frac{n+1}{2}\,\Gamma\left(1-\frac{n+1}{2}\right)}{2}$$

$$= \frac{\pi}{2\sin\left(\frac{n+1}{2}\right)\pi}$$

$$= \frac{\pi}{2}\sin\left(\frac{\pi}{2} + n\frac{\pi}{2}\right)$$

$$= \frac{\pi}{2}\cos n\frac{\pi}{2}$$

$$= \frac{\pi}{2}\sec n\frac{\pi}{2}$$

Ques: $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \pi$

$$\frac{\Gamma 3/4\ \Gamma 1/2}{2\Gamma\frac{\frac{1}{2}+0+2}{2}} \times \frac{\Gamma 1/4\ \Gamma 1/2}{2\Gamma\frac{-\frac{1}{2}+0+2}{2}}$$

$$\frac{\Gamma\frac{3}{4}\,\Gamma\frac{1}{2}}{2\Gamma 5/4} \times \frac{\Gamma\frac{1}{4}\,\Gamma\frac{1}{2}}{2\Gamma 3/4}$$

$$\frac{\frac{1}{4}\sqrt{\pi}}{2\times\frac{1}{4}\times\sqrt{\frac{1}{4}}} \times \frac{\frac{3}{4}\sqrt{\pi}}{2\sqrt{\frac{3}{4}}}$$

Qu:- $$\int_0^1 \frac{x^2}{(1-x^4)^{1/2}}dx \times \int_0^1 \frac{dx}{(1+x^4)^{1/2}} = \frac{\pi}{4\sqrt{2}}$$

Let $x^2 = \sin\theta$

$2x\,dx = \cos\theta\,d\theta$

$$= \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta}\frac{\cos\theta}{2\sqrt{\sin\theta}}d\theta \times \int_0^{\pi/4} \frac{\sec^2\theta\,d\theta}{2\sqrt{\tan\theta}\,2\sec\theta}$$

$$\frac{1}{2}\int_0^{\pi/2} \sqrt{\sin\theta}\,d\theta \times \int_0^{\pi/4} \frac{d\theta}{2\sqrt{\sin\theta\cos\theta}}$$

$$\frac{1}{2}\int_0^{\pi/2} \sqrt{\sin\theta}\cdot d\theta \times \int_0^{\pi/4} \frac{d\theta}{\sqrt{2}\sqrt{\sin 2\theta}}$$

Let $2\theta = t$

$2d\theta = dt$

$$\frac{1}{2}\int_0^{\pi/2} \sqrt{\sin\theta}\,d\theta \times \int_0^{\pi/2} \frac{dt}{2\sqrt{2}\sqrt{\sin t}}$$

$$\frac{1}{2}\left[\frac{\Gamma\frac{3}{4}\,\Gamma\frac{1}{2}}{2\Gamma\frac{5}{4}}\right] \times \frac{1}{2\sqrt{2}} \quad \frac{\Gamma\frac{1}{4}\,\Gamma\frac{1}{2}}{2\Gamma\frac{3}{4}}$$

$$\frac{1}{4}\frac{\sqrt{\pi}}{2\times\frac{1}{4}\times\sqrt{\frac{1}{4}}} \times \frac{\Gamma\frac{1}{4}\,\sqrt{\pi}}{4\sqrt{2}} = \frac{\pi}{4\sqrt{2}}$$

Ques: $\Gamma\frac{1}{4}\Gamma\frac{3}{4} = 2\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = 4\int_0^{\infty} \frac{x^2 dx}{1+x^4} = \pi\sqrt{2}$

$$4\int_0^{\infty} \frac{x^2 dx}{1+x^4}$$

Let $x^2 = \tan\theta$

$2x\,dx = \sec^2\theta\, d\theta$

$$4\int_0^{\pi/2} \frac{\sqrt{\tan\theta} \times \sec^2\theta\, d\theta}{2 \times \sec^2\theta}$$

$$2\int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta\, d\theta$$

$$2\left[\frac{\Gamma 3/4\ \Gamma 1/4}{2\Gamma\frac{1/2 - 1/2 + 2}{2}}\right]$$

$$= 2\left[\frac{\Gamma\frac{3}{4}\Gamma\frac{1}{4}}{2}\right]$$

$$\frac{1}{4}\Gamma\frac{1}{4}$$

$$= \frac{\pi}{\sin\frac{1}{4}\pi} = \pi\sqrt{2}$$

Ques: $2^n \Gamma\frac{n+1}{2} = 1 \cdot 3 \cdot 5 \cdots (2n+1)\sqrt{\pi}$

$2^n \sqrt{n+\frac{1}{2}} = 2^n \left(n+\frac{1}{2}\right) \sqrt{n+\frac{1}{2}}$

$2^n \left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right) \sqrt{n+\frac{3}{2}}$

$2^n \left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right) \cdots\cdots \frac{1}{2}\sqrt{\frac{1}{2}}$

$\frac{2^n (2n-1)(2n-3)\cdots\cdots 1 \cdot \sqrt{\pi}}{2^n}$

Ques: $\sqrt{\frac{3}{2}-x}\sqrt{\frac{3}{2}+x} = \left(\frac{1}{4}-x^2\right)\pi \sec \pi x$.

$\left(\frac{1}{2}-x\right)\sqrt{\frac{1}{2}-x}\left(\frac{1}{2}+x\right)\sqrt{\frac{1}{2}+x}$

$\left(\frac{1}{4}-x^2\right)\sqrt{\frac{1}{2}-x^0}\sqrt{1-\left(\frac{1}{2}-x\right)}$

$\left(\frac{1}{4}-x^2\right)\left(\frac{\pi}{\sin\left(\frac{1}{2}-x\right)\pi.}\right)$

$\left(\frac{1}{4}-x^2\right)\frac{\pi}{\sin\left(\frac{\pi}{2}-x\pi\right)} \Rightarrow \left(\frac{1}{4}-x^2\right)\pi \cdot \sec \pi x$

Imp ⇒ duplication
wi elder formula proof

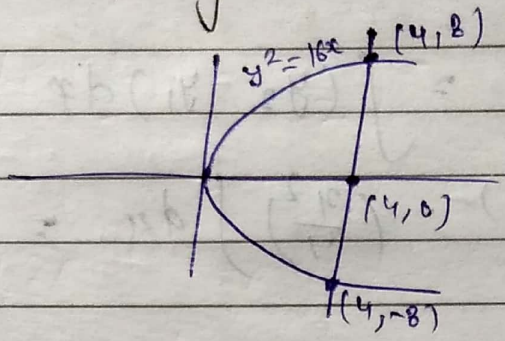
$\Gamma\frac{1}{2}$ ⇒ Prof. / Relation b/w β and Γ.

Ques: $\int_0^1 \frac{dx}{(1-x^4)^{1/2}} = \frac{\Gamma 2}{8\sqrt{\pi}} \Gamma\frac{1}{4}^2$

21/11/18

Area of Bounded region by definate Integrals

Qu: find the area of the region bounded by $x = 4$ & $y^2 = 16x$



$$= 2\int y\,dx = 2\int_0^4 \sqrt{16x}\,dx = 8\int_0^4 x^{1/2}\,dx$$

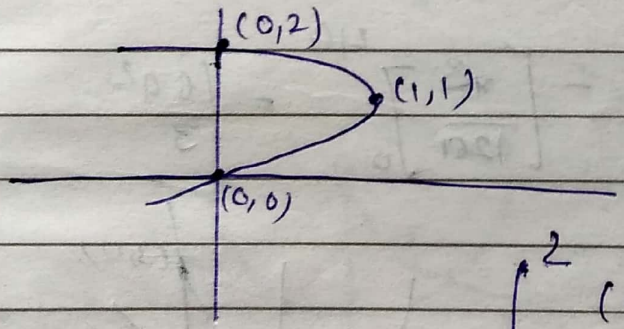
$$= \frac{8\times 2}{3}\left[x^{3/2}\right]_0^4 = \frac{16}{3}\left[4^{3/2} - 0\right]$$

$$= \frac{16}{3}\times 8 = \frac{128}{3}$$

Qu $y^2 = 2y - x$

at

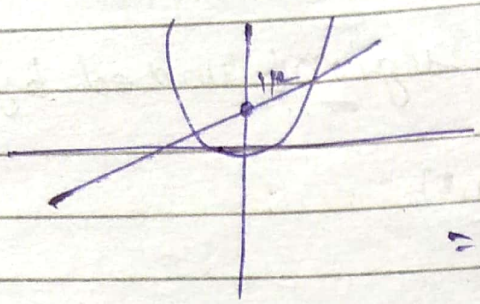
x	0	0	1
y	2	0	1



$$\int x\,dy$$

$$\int_0^2 (2y - y^2)\,dy = \frac{4}{3}$$

Qu: $x^2 = 4y$ & $x = 4y - 2$

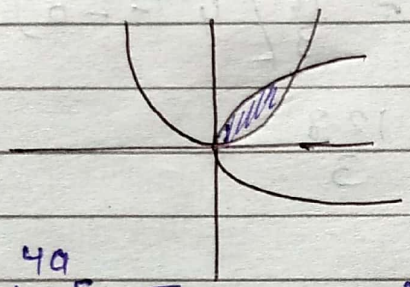


x	0	0	1
y	1/2	-2	3/4

$$= \int (y_2 - y_1)\, dx$$

$$= \int_{-1}^{2} \left(\frac{(x+2)}{2} - \left(\frac{x^2}{4}\right) \right) dx = \frac{9}{8}$$

Qu: Find the area b/w the parabolas $y^2 = 4ax$, $x^2 = 4ay$



$$= \int (y_2 - y_1)\, dx$$

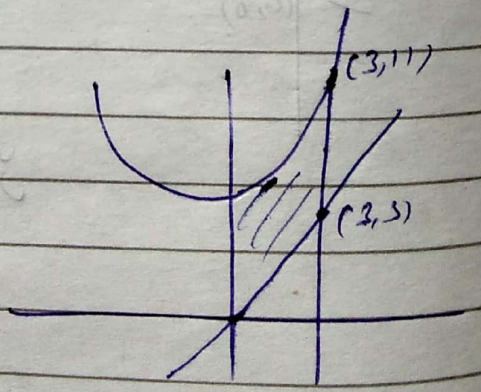
$$= \int_0^{4a} \left[\sqrt{4ax} - \frac{x^2}{4a} \right] dx$$

$$= \sqrt{4a} \left[\frac{2x^{3/2}}{3} \right]_0^{4a} - \left[\frac{x^3}{12a} \right]_0^{4a} = \frac{16a^2}{3}$$

Qu: $y = x^2 + 2$

$y = x$

$x = 0$, $x = 3$

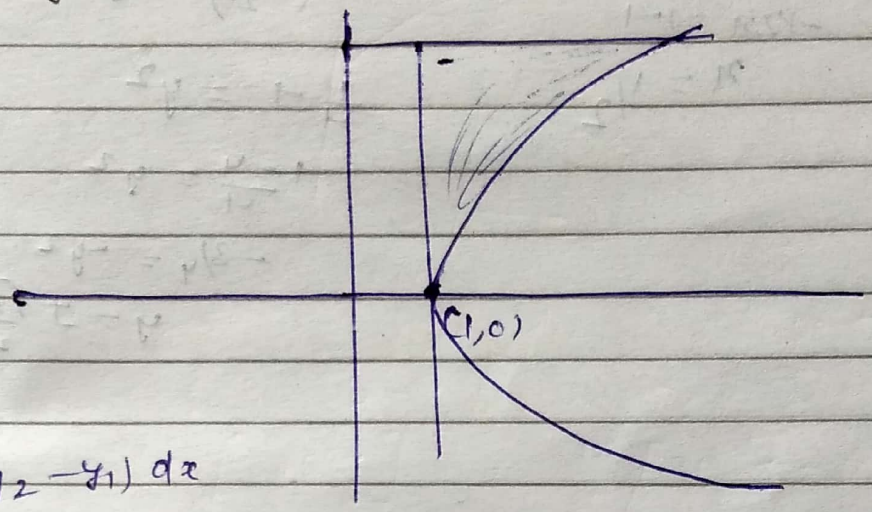


$$\int (y_2 - y_1)\, dx$$

$$\int_0^3 (x^2+2) - x\,dx$$

$$\left[\frac{x^3}{3} + 2x - \frac{x^2}{2}\right]_0^3 = \frac{21}{2}$$

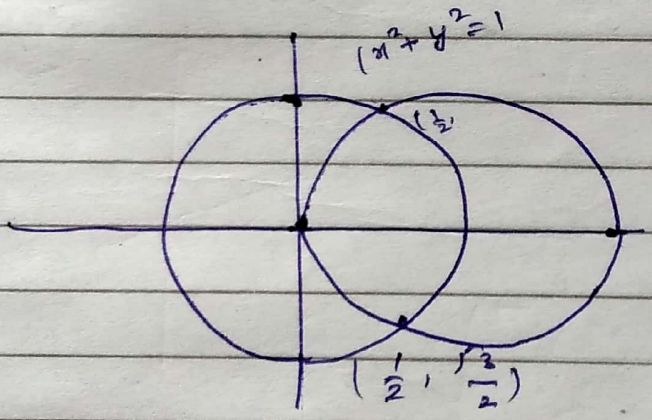
Que: $y^2 = 4a^2(x-1)$ & the line $x=1, y=4a$



$$\int (y_2 - y_1)\,dx$$

$$\int_1^5 (4a - 2a\sqrt{x-1})\,dx$$

Que: $x^2 + y^2 = 1$ & $(x-1)^2 + y^2 = 1$



$$2\int y\,dx$$

$$2\left[\int_0^{1/2} \sqrt{1-(x-1)^2}\,dx + \int_{1/2}^{1} \sqrt{1-x^2}\,dx\right.$$

$(x-1)^2 + y^2 = 1$

$x^2 + 1^2 - 2x + y^2 = 1$

$1 + 1^2 - 2x = 1$

$2 - 2x = 1$

$-2x = 1 - 2$

$-2x = 1 - 1$

$x = 1/2$

$(x-1)^2 + y^2 = 1$

$\left(\frac{1}{2} - 1\right)^2 + y^2 = 1$

$\left(\frac{1-2}{2}\right)^2 + y^2 = 1$

$\left(\frac{-1}{2}\right)^2 + y^2 = 1$

$\frac{1}{4} - 1 = y^2$

$1 - \frac{4}{4} = y^2$

$-3/4 = -y^2$

$y = \pm \frac{\sqrt{3}}{2}$