Unit -

05/09/18

Infinite Series

A series $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ is called Infinite Series.

* Convergent series:

If $\lim_{n \to \infty} S_n$ = finite

S_n = sum of first n terms.

then we can say this series is Convergent

(ii) If $\lim_{n \to \infty} S_n = \infty$ or $-\infty$

then we say this series is diversent

(iii) oscillatory series:

If $\lim_{n \to \infty} S_n$ = neither finite nor infinite

$(1 - 1 + 1 - 1 - \cdots)$

$S_n = 0$ (n is even)

$= 1$ (n is odd)

$\log \infty =$

$e^{\infty} = \infty$

$e^{-\infty} =$

(1) (2) $\left(\frac{1}{2}\right)^{\infty} \to 0$ $\left(\frac{1}{2}\right)^{\infty} = 0$ 53

Q. $1 + 2 + 3 + 4 + \cdots\cdots$

$S_n = \frac{n(n+1)}{2}$

$\lim_{n \to \infty} S_n = \infty$

given series is ~~convergent~~ divergent

(2) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\cdots$

$S_n = \frac{a(1-r^n)}{(1-r)} = \frac{(1-(\frac{1}{2})^n}{(1-\frac{1}{2})}$

$\lim_{n \to \infty} S_n = \frac{1}{1/2} = 2$ Convergent

Q. $\log 3 + \log \frac{4}{3} + \log \frac{5}{4} + \cdots\cdots$

$\log \left(3 \cdot \frac{4}{3} \cdot \frac{5}{4} \right)$

$\log 3 + \log \frac{4}{3} + \log \frac{5}{4} + \cdots - \log \left(\frac{n+3}{n+2} \right)$

$\log \left(3 \cdot \frac{4}{3} \cdot \frac{5}{4} \cdots \frac{n+3}{n+2} \right)$

$\log (n+3)$

$\lim_{n \to \infty} S_n = \infty$ (divergent)

Proof Sub

* Property of GP (Geometry):

The series $1 + r + r^2 + \cdots$

(i) is convergent if $|r| < 1$

(ii) divergent if $r \geq 1$

(iii) Oscillatory if $r \leq -1$

$$S_n = \frac{(1 - r^n)}{(1 - r)}$$

Sub:

(i) if $|r| < 1$

$$\lim_{n \to \infty} r^n = 0$$

$$\lim_{n \to \infty} S_n = \frac{1}{1 - r} = \text{finite.}$$

given series is Convergent if $\boxed{|r| < 1}$

Sub:

(ii) if $r \geq 1$

$$\lim_{n \to \infty} r^n = \infty$$

$$\lim_{n \to \infty} S_n = -\infty$$

the given series is divergent.

(ii) when $r = 1$

Series $1 + 1 + 1 + 1 + 1$

$S_n = n$

$\lim_{n \to \infty} S_n = \infty$

series is divergent

if $r \geq 1$

(iii) if $r = -1$

$1 - 1 + 1 - 1 + - - - -$

$S_n = 0$ if n is even
1 if n is odd.

Hence the value of S_n is not unique.

∴ the given series is oscillatory

Let $r = -k$

$r^n = (-1)^n (k)^n$.

$S_n = \frac{1 - r^n}{1 - r} = \frac{1 - (-1)^n k^n}{1 + k}$

$\lim_{n \to \infty} S_n = -\infty, \infty$

$\lim_{n\to\infty} S_n = -\infty$ if n = even

$= \infty$ if n = odd

not unique.

Hence Seq/g Series is Oscillatory

(2*) **Cauchy's fundamental test:**

only divergent

i.f. = 0 fail.

If $\lim_{n\to\infty} u_n \neq 0$

then the given ~~infinit~~ series is divergent

Q. $\sqrt{\frac{1}{4}} + \sqrt{\frac{2}{9}} + \sqrt{\frac{3}{8}} - - -$

Test of the series

Sol^n: $u_n = \sqrt{\frac{n}{2(n+1)}}$

$\lim_{n\to\infty} u_n = \sqrt{\frac{1}{2+\frac{2}{n}}}$

$\lim_{n\to\infty} u_n = \frac{1}{\sqrt{2}} \neq 0$

Hence by Cauchy's fund. test

Say given series is divergent

Q. $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$

$u_n = \cos\frac{1}{n}$

$\lim_{n\to\infty} u_n = \cos 0$

$= 1 \neq 0$

by Cauchy fund. test ie. divergent

Q. $\sum_{n=1}^{\infty} \sqrt{\frac{n}{n+1}}$

$u_n = \sqrt{\frac{n}{n+1}}$

$\lim_{n\to\infty} u_n = 1 \neq 0$

by Cauchy fund test divergent

Q. $\sum_{n=1}^{\infty} \sin\frac{1}{n}$

$u_n = \sin\frac{1}{n}$

$= \left[\frac{1}{n} - \frac{1}{n^3 3!} + \frac{1}{n^5 5!} - \cdots\right]$

$\lim_{n\to\infty} u_n = 0$ Can't say

[fail the test]

(3) P series test :-

the series $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots\cdots$

is convergent if $p > 1$
divergent if $p \leq 1$

(4) Comparision test :-

if Two positive term series Σu_n & Σv_n be such that

$$\lim_{n \to \infty} \frac{u_n}{v_n} = k \quad (\text{finite \& non-zero})$$

then

Σu_n is convergent
if Σv_n is convergent

Σu_n is divergent
if Σv_n is divergent

[applied either numerator and denominator are contain n^{th} term.]

Ex: $\Sigma u_n = \Sigma \sin\left(\frac{1}{n}\right)$

$$= \left(\frac{1}{n} - \frac{1}{n^3 3!} + \frac{1}{n^5 5!} - \cdots\right)$$

$$= \frac{1}{n}\left(1 - \frac{1}{n^2 3!} + \frac{1}{n^4 5!} - \cdots\right)$$

$$u_n = \frac{1}{n}$$

$$\frac{u_n}{v_n} = \left[1 - \frac{1}{n^2 3!} + \frac{1}{n^4 5!} - \cdots\right]$$

$$\lim_{n \to \infty} \frac{u_n}{v_n} = 1 \neq 0 \text{ & finite.}$$

$$v_n = \frac{1}{n}$$

$\lim_{n \to \infty} v_n =$ 0 $\therefore \Sigma v_n$ is divergent by p test.

$\therefore p = 1$

Hence $\therefore$ by Comp. test

Σu_n is divergent

Q. Test the convergence of Series:

$$\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots$$

$$u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}} = \frac{1}{\sqrt{n}\left(1+\sqrt{1+\frac{1}{n}}\right)}$$

$$v_n = \frac{1}{\sqrt{n}}$$

$$\frac{u_n}{v_n} = \frac{1}{1+\sqrt{1+\frac{1}{n}}}$$

≤1 - div
>1 - conv

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{2} \neq 0 \text{ \& finite}$$

$$\Sigma v_n = \frac{1}{\sqrt{n}}$$

∴ by P test Σv_n is divergent

$$P = \frac{1}{2} < 1$$

∴ by Comp. test Σv_n is divergent

Q. Check

$$\Sigma\left(3\sqrt{n^3+1} - n\right)$$

Sol"

$$u_n = \frac{\left(3\sqrt{n^3+1} - n\right) \times \left(3\sqrt{n^3+1} + n\right)}{3\sqrt{n^3+1} + n)}$$

$$= \frac{(n^3+1)^{2/3} - n^2}{(n^3+1)^{1/3} + n}$$

$$u_n = \frac{n^2\left(1 + \frac{1}{n^3}\right)^{2/3} - 1)}{n\left(1 + \frac{1}{n^3}\right)^{1/3} + 1)}$$

$$u_n = n$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = 0$$

Hence Comparision test is fail

Q. test $\sum \frac{2n^2+3n}{5+n^5}$

$$u_n = \frac{2n^2+3n}{5+n5}$$

$$= \frac{n^2(2+3/n)}{n^5(1+5/n5)}$$

$$v_n = \frac{1}{n^3}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left(\frac{2+3/n}{1+5/n^5}\right) = 2 \neq 0$$

$$v_n = \frac{1}{n^3}$$

∴ by Test $\sum v_n$ is convergent

∵ $p = 3 > 1$

∴ by Compaision test $\sum u_n$ is Convergent

Q. $\frac{1}{1\cdot2\cdot3} + \frac{3}{2\cdot3\cdot4} + \frac{5}{3\cdot4\cdot5} \cdots\cdots$

$a+(n-1)d$
$1+(n-1)2$

$$u_n = \frac{(2n-1)}{n(n+1)(n+2)}$$

$$= \frac{n(2-\frac{1}{n})}{n^3(1+\frac{1}{n})(1+\frac{2}{n})}$$

$$v_n = \frac{1}{n^2}$$

$$\frac{u_n}{v_n} = \frac{\left(2 - \frac{1}{n}\right)}{\left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right)}$$

$$\lim_{n \to \infty} \frac{u_n}{v_n} = 2 \neq 0 \text{ & finite}$$

$\therefore$ by P test $\quad p = 2 > 1$

$\sum v_n$ Convergent

Hence u_n is convergent

Qu: $\sum \frac{3\sqrt{3n^2+1}}{\sqrt[4]{4n^3+2n+7}}$

$$u_n = \frac{(3n^2+1)^{1/3}}{(4n^3+2n+7)^{1/4}}$$

$$u_n = \frac{n^{2/3}}{n^{3/4}} \frac{\left[3 + \frac{1}{n^2}\right]^{1/3}}{\left[4 + \frac{2}{n^2} + \frac{7}{n^3}\right]^{1/4}}$$

$$= \frac{1}{n^{1/12}} \frac{\left[3 + \frac{1}{n^2}\right]^{1/3}}{\left[4 + \frac{2}{n^2} + \frac{7}{n^3}\right]^{1/4}}$$

$$u_n = \frac{1}{n^{1/12}}$$

$$\lim_{n \to \infty} \frac{u_n}{v_n} = \frac{3^{1/3}}{4^{1/4}} \neq 0 \text{ & finite}$$

Comparision test hold.

$\therefore$ by P test $\quad \sum v_n = \sum \frac{1}{n^{1/2}}$ is divergent

$\therefore$ by Comp. test $\sum v_n$ is divergent.

Q. Find the convergence of series whose n^{th} term is $n^{\log x}$

$$S_n = n^{\log x}$$
$$= 1^{\log n} + 2^{\log n} + 3^{\log n} + \cdots$$
$$= \frac{1}{1^{-\log n}} + \frac{1}{2^{-\log n}} + \frac{1}{3^{-\log n}} + \cdots$$

let $-\log x = p$

$\therefore$ By P test

$\frac{1}{n^p}$ is convergent if $p > 1 \Rightarrow -\log n > 1$

$\log\left(\frac{1}{n}\right) > 1$

$\log\left(\frac{1}{n}\right) > \log_e e \qquad \frac{1}{n} > e$

$n < \frac{1}{e}$

$\frac{1}{n^p}$ is divergent if $p \leq 1$

$-\log n \leq 1$

$\log\left(\frac{1}{n}\right) \leq 1$

$\frac{1}{n} \leq e$

$n \geq \frac{1}{e}$

Q. $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} - \cdots$

$u_n = \frac{n+1}{n^p}$

$= \frac{n(1+1/n)}{n^p} = \frac{n}{n^p}\left(1+\frac{1}{n}\right)$

$v_n = \frac{1}{n^{p-1}}$

$\frac{u_n}{v_n} = \left(1+\frac{1}{n}\right)$

$\lim_{n\to\infty} \frac{u_n}{v_n} = 1 \neq 0$ & finite

$v_n = \frac{1}{n^{p-1}}$ by p test $\sum \frac{1}{n^{p-1}}$ is

Convergent if $p-1 > 1$

$p > 2$

and $\sum u_n$ is dugt if $p-1 \leq 1$

$p \leq 2$

$\therefore$ by Comp. test

$\sum u_n$ is cgt if $p > 2$

& dgt if $p \leq 2$

Q: $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \cdots$ $\frac{n^2}{(n+1)^2}$

$$u_n = \frac{n^n}{(n+1)^{(n+1)}}$$

$$u_n = \frac{n^n}{n^{n+1}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n}$$

$$v_n = \frac{1}{n}$$

$$\frac{u_n}{v_n} = \frac{1}{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{e} \neq 0 \text{ \& finite}$$

Comparision test hold

$\therefore$ by p test $\sum v_n = \sum \frac{1}{n}$ is dgt $p = 1$

$\therefore$ by Comp. test $\sum u_n$ is divergent

* D'Alembert Ratio test :

If $\sum u_n$ is Positive term series such that

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = K.$$

where K is finite

then series is convergent if $K > 1$

divergent if $K < 1$

test fail if $K = 1$

Ex :- $\sum \frac{n!}{n^n}$

$u_n = \frac{n!}{n^n}$, $u_{n+1} = \frac{(n+1)!}{(n+1)^{(n+1)}}$

$$\frac{u_n}{u_{n+1}} = \frac{n!}{n^n} \times \frac{(n+1)^{n+1}}{(n+1)!}$$

$$= \frac{(n+1)^{n+1}}{n^n (n+1)}$$

$$= \frac{n^{n+1}\left(1+\frac{1}{n}\right)^{n+1}}{n^{n+1}\left(1+\frac{1}{n}\right)}$$

$$= e > 1$$

2. -

$\therefore$ by DAR test $\sum u_n$ is convergent

Q. $\sum \frac{n^2}{3^n}$

$$u_n = \frac{n^2}{3^n}$$

$$u_{n+1} = \frac{(n+1)^2}{3^{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^2}{3^n} \times \frac{3^{n+1}}{(n+1)^2}$$

$$= \frac{3}{(1+\frac{1}{n})^2}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 3 > 1$$

$\therefore$ DAR test it is convergent.

Q. $\sum \frac{x^{n+1}}{n3^n}$

$$u_n = \frac{x^{n+1}}{n3^n}, \quad u_n = \frac{x^n}{(n+1)3^{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{x^{n+1}}{n3^n} \times \frac{(n+1)3^{n+1}}{x^n}$$

$$= \frac{1}{n}(1+\frac{1}{n})3$$

$$= \frac{3}{n}$$

if $\frac{3}{n} > 1 \Rightarrow x < 3$ $\sum u_n$ is cgt.

$\frac{3}{n} < 1 \Rightarrow x > 3$ $\sum u_n$ is divergent

~~ratio~~ test fails $\frac{3}{n} = 1 \Rightarrow n = 3$

If $x = 3$

$$u_n = \frac{3^{n-1}}{n\,3^n}$$

$$u_n = \frac{1}{3n} = \frac{1}{3(n)}$$

$$\Sigma u_n = \frac{1}{3} + \frac{1}{6} + \frac{1}{9} - - - - \quad (\text{H.P})$$

$$v_n = \frac{1}{n}$$

$$\frac{u_n}{v_n} = \frac{1}{3} \neq 0 \text{ & finite}$$

$\therefore$ by P test $\Sigma u_n = \Sigma \frac{1}{n}$ is divergent $\because P = 1$

" Comp test Σu_n is divergent if $x = 3$

Convergent if $x < 3$
divergent if $x \geq 3$

Q:- $$1 + \frac{\alpha+1}{\beta+1} + \frac{(\alpha+1)(2\alpha+1)}{(\beta+1)(2\beta+1)} + \frac{(\alpha+1)(2\alpha+1)(3\alpha+1)}{(\beta+1)(2\beta+1)(3\beta+1)} + \cdots$$

$$u_n = \frac{(\alpha+1)(2\alpha+1) - - - (n\alpha+1)}{(\beta+1)(2\beta+1) - - - (n\beta+1)}$$

$$u_{n+1} = \frac{(\alpha+1)(2\alpha+1) - - (n\alpha+1)(n+1)\alpha+1)}{(\beta+1)(2\beta+1) - - (n\beta+1)(n+1)\beta+1)}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)\beta + 1}{(n+1)\alpha + 1}$$

$$= \frac{\left(1+\frac{1}{n}\right)\beta + \frac{1}{n}}{\left(1+\frac{1}{n}\right)\alpha + \frac{1}{n}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{\beta}{\alpha}$$

if $\frac{\beta}{\alpha} > 1 \Rightarrow \beta > \alpha$ then by DAR test

$\sum u_n$ is cgt

if $\frac{\beta}{\alpha} < 1 \Rightarrow \beta < \alpha$

$\sum u_n$ is dgt

If $\frac{\beta}{\alpha} = 1 \Rightarrow \beta = \alpha$ test fail

If $\beta = \alpha$

$u_n = 1$

$\sum u_n = 1 + 1 + 1 + 1 - \cdots$

$S_n = n$

$\lim_{n\to\infty} S_n = \infty$

$\therefore$ Symm test $\sum u_n$ is dgt if $\alpha = \beta$

Q. $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} \cdots$

$$u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}$$

$$u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{x^{2n-2}}{(n+1)\sqrt{n}} \times \frac{(n+2)\sqrt{n+1}}{x^{2n}}$$

$$\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{2}{n}\right)\sqrt{1+\frac{1}{n}}}{\left(1+\frac{1}{n}\right)x^2}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x^2}$$

if $\frac{1}{x^2} > 1 \Rightarrow x^2 < 1$ by DART

$\sum u_n$ is cgt

if $\frac{1}{x^2} < 1 \Rightarrow x^2 > 1$ by DART

$\sum u_n$ is dgt

if $\frac{1}{x^2} = 1 \Rightarrow x^2 = 1$

D.A.R.T fail

$$a_n = \frac{1}{(n+1)\sqrt{n}}$$

$$= \frac{1}{n^{3/2}\left(1+\frac{1}{n}\right)}$$

$$u_n = \frac{1}{n^{3/2}}$$

$$\frac{a_n}{u_n} = \frac{1}{1+\frac{1}{n}}$$

$$\lim_{n\to\infty} \frac{a_n}{u_n} = 1 \neq 0 \text{ \& finite}$$

$\therefore$ by f test $\sum u_n = \sum \frac{1}{n^{3/2}}$ is cgt $\therefore$ by Comp. test 1

$\sum a_n = \frac{1}{(n+1)\sqrt{n}}$ is cgt

$\sum u_n$ is cgt if $n^2 > 1$

if $n^2 \leq$ by DMRT $\sum$ cgt

Q. $\frac{1}{1+2} + \frac{2}{1+2^2} + \frac{3}{1+2^3} + \cdots$

$$u_n = \frac{n}{1+2^n}$$

$$u_{n+1} = \frac{n+1}{1+2^{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{n}{1+2^n} \times \frac{1+2^{n+1}}{n+1}$$

$$= \frac{1}{\left(1+\frac{1}{2n}\right)} \times \frac{\left(2+\frac{1}{2n}\right)}{\left(1+\frac{1}{n}\right)}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 2 > 1$$

Hence by D.A.R Test $\sum u_n$ = Convergent

Que! $1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + \cdots$

$$u_n = \left(\frac{2^n - 2}{2^n + 1}\right) x^{n-1}$$

Jail

Que: $u_n = \frac{x^n}{a+\sqrt{n}}$

$u_n = x^n$ (

$u_{n+1} = \frac{x^{n+1}}{a+\sqrt{n+1}}$

$\frac{u_n}{u_{n+1}} = \frac{x^n}{a+\sqrt{n}} \times \frac{a+\sqrt{n+1}}{x^{n+1}}$

$= \frac{\frac{a}{\sqrt{n}} + \sqrt{1+\frac{1}{n}}}{\frac{a}{\sqrt{n}}+1} \times x$

$n-\frac{1}{2}$

$\frac{2n-1}{2}$

Limit $n \to \infty$ $\frac{u_n}{u_{n+1}} = \frac{1}{n}$

$\frac{1}{n} \to 1$

$\frac{1}{n} < 1$

$\frac{1}{n} = 1$

$\frac{1}{n} = 1$ $n = 1$ so, DAR test is fail.

So, $u_n = \frac{x^n}{a+\sqrt{n}} = \frac{x^n}{n^{1/2}} \left(\frac{1}{\frac{a}{\sqrt{n}}+1}\right)$

$u_n = x^{\frac{2n-1}{2}}$ $v_n = \frac{1}{\sqrt{n}}$

Limit $n \to \infty$ $\frac{u_n}{v_n} =$

For comparision test

Q. $\sum \frac{3n+1}{4n+3}$ (n)

$u_n = \frac{3n+1}{4n+3}$ $\quad u_{n+1} = \frac{3(n+1)+1}{4(n+1)+3}$ (n)

$u_{n+1} = \frac{3n+4}{4n+7}$

Series is divergent

$$\frac{u_n}{u_{n+1}} = \frac{1}{n}\left(\frac{3n+1}{4n+3} \times \frac{4n+7}{3n+4}\right)$$

$$\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{1}{3n}\right)\left(1+\frac{7}{4n}\right)}{\left(1+\frac{3}{4n}\right)\left(1+\frac{4}{3n}\right)}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{n}$$

* when dAR Test fail we will apply next test = Raabes's test.

* Raabes's test:

If $\sum u_n$ is positive term series such that

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = K$$

then the series is convergent if $K > 1$
divergent if $K < 1$

if $K = 1$ Raabes test fail.

Q. $\sum \frac{1}{\sqrt{n+1}-1}$

Both apply
① Comparison test
② Raabe's test

$$u_n = \frac{1}{\sqrt{n+1}-1}$$

$$u_{n+1} = \frac{1}{\sqrt{n+2}-1}$$

$$\frac{u_n}{u_{n+1}} = \frac{1}{\sqrt{n+1}-1} \times \frac{\sqrt{n+2}-1}{1}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{\sqrt{1+\frac{2}{n}}-\frac{1}{\sqrt{n}}}{\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}} = 1$$

D'Alembert test fail.

Hence we will apply Raabe's test

$$\frac{u_n}{u_{n+1}} + = \left[\frac{\sqrt{n+2}-1}{\sqrt{n+1}-1} - 1\right]$$

$$= \frac{\sqrt{n+2}-1-\sqrt{n+1}+1}{\sqrt{n+1}-1}$$

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{n\left(\sqrt{n+2}-\sqrt{n+1}\right)}{\sqrt{n+1}-1}$$

$$= \frac{n^{3/2}\left(\sqrt{1+\frac{2}{n}} - \sqrt{1-\frac{1}{n}}\right)}{n^{1/2}\left(\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}}\right)}$$

$$= 0$$

$\therefore$ divergent.

Q. $\frac{1}{2} + \frac{1\cdot3}{2\cdot4} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}$

$$u_n = \frac{1\cdot3\cdot5 \cdots (2n-1)}{2\cdot4\cdot6 \cdots 2n} \qquad u_n = \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots 2n}$$

$$u_{n+1} = \frac{1\cdot3\cdot5 \cdots (2n-1)(2n+1)}{2\cdot4\cdot6 \cdots (2n)(2n+2)}$$

$$\frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1$$ Hence DAR test fail

$\therefore$ by R test

$$\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{(2n+2)-(2n+1)}{(2n+1)}$$

$$= \frac{1}{2n+1}$$

x^{2n-1}

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{n}{2n+1}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{1}{2} < 1$$

$\therefore$ by R test $\sum u_n$ is dgt

Q. find out the n^{th} term.

$$\frac{x}{1} + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\times 3}{2\times 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} \cdots$$

$$u_n = \frac{1\cdot 3\cdot 5 \cdots (2n-1)\, x^{2n-1}}{(2\cdot 4\cdot 6 \cdots 2n)(2n+1)}$$

$$u_{n+1} = \frac{1\cdot 3\cdot 5 \cdots (2n-1)(2n+1)\, x^{2n+1}}{2\cdot 4\cdot 6 \cdots (2n)(2n+2)(2n+1)}$$

$$\frac{u_n}{u_{n+1}} = \frac{2n+2}{x^2(2n+1)}$$

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)}{(2n-1)x^2}$$

$$\lim \frac{u_n}{u_{n+1}} = \frac{1}{x^2}$$

if $\frac{1}{x^2} > 1 \Rightarrow x^2 < 1 \Rightarrow$ cgt

if $\frac{1}{x^2} < 1 \Rightarrow x^2 > 1 \Rightarrow$ dgt.

if $x^2 = 1$

D'Alembert test fail.

Hence we apply Raabe's test

$$\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{2n+2-(2n-1)}{(2n+1)}$$

$$= \frac{3}{(2n+1)}$$

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{3n}{2n+1}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{3}{2} > 1$$

by R. test

$\sum u_n$ is cgt if $x^2 = 1$

* Logarithmic test : (value depends on e, D'Alembert test fail)

if $\sum u_n$ is a positive term series such that $\lim_{n\to\infty} \left(n \log \frac{u_n}{u_{n+1}}\right) = K$

then $\sum u_n$ is cgt if $K > 1$

" " dgt if $K < 1$

$K = 1$ test fails

Q. $x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!}$

$$u_n = \frac{n^n x^n}{n!}, \quad u_{n+1} = \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^n x^n}{n!} \cdot \frac{(n+1)!}{(n+1)^{(n+1)} x^{n+1}}$$

$$= \frac{n^n (n+1)}{n^{n+1} (1+\frac{1}{n})^n (1+\frac{1}{n}) x}$$

$$= \frac{u_n}{u_{n+1}} = \frac{(1+1/n)}{(1+\frac{1}{n})^n \cdot (1+\frac{1}{n}) x}$$

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \frac{1}{ex}$$

if $\frac{1}{ex} > 1 \Rightarrow x < \frac{1}{e} \Rightarrow \sum u_n$ (is) - is cgt

If $\frac{1}{ex} < 1 \Rightarrow x > \frac{1}{e} \Rightarrow \sum u_n$ is dgt

If $\frac{1}{ex} = 1 \Rightarrow x = \frac{1}{e}$ test fail

at $x = \frac{1}{e}$

$$\frac{u_n}{u_{n+1}} = \frac{e}{(1+\frac{1}{n})^n}$$

$$\log\left(\frac{u_n}{u_{n+1}}\right) = \log\left(\frac{e}{(1+\frac{1}{n})^n}\right)$$

$$= \log e - n\log\left(1+\frac{1}{n}\right)$$

$$= 1 - n\left[n\left(\frac{1}{n} - \frac{1}{n^2 \cdot 2} + \frac{1}{n^3 \cdot 3} - \cdots\right)\right]$$

~~$\log(1+$~~

$$\log\left(\frac{u_n}{u_{n+1}}\right) \left[1 - 1 + \frac{1}{2n} - \frac{3}{n^2} \cdots\right]$$

$$n\log\left(\frac{u_n}{u_{n+1}}\right) = \left[\frac{1}{2} - \frac{3}{n} + \cdots\right]$$

$$\lim_{n\to\infty} n\log\left(\frac{u_n}{u_{n+1}}\right) = \frac{1}{2} < 1$$

∴ By log test $\sum u_n$ is dgt at $x = 1/e$

$$0 = 1 + \frac{2x}{2} + \frac{3^2 x^2}{3!} \cdots$$

$$u_n = \frac{n^{n-1} x^{n+1}}{n!}$$

$$u_{n+1} = \frac{(n+1)^n x^n}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^{n-1} x^{n+1}}{n!} \times \frac{(n+1)!}{(n+1)^n (x)^n}$$

$$= \frac{(n+1)}{n\left(1+\frac{1}{n}\right)^n x}$$

$$= \frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)^n x}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{e^x}$$

If $\frac{1}{e^x} > 1 \Rightarrow x < \frac{1}{e} \Rightarrow \Sigma u_n$ is cgt

If $\frac{1}{e^x} < 1 \Rightarrow x > \frac{1}{e} \Rightarrow \Sigma u_n$ is dgt

If $\frac{1}{e^x} = 1 \Rightarrow x = \frac{1}{e}$ Test fails.

∴ by log test at $x = \frac{1}{e}$

$$\log \frac{u_n}{u_{n+1}} = \log \left(\frac{(1+1/n)}{(1+\frac{1}{n})^n}\right) e \quad (1)$$

$$= \log e^e + \log\left(1+\frac{1}{n}\right) - n \log\left(1+\frac{1}{n}\right)$$

$$1 + \left(\frac{1}{n} + \frac{1}{n^2 2} + \frac{1}{n^3 3} - \cdots\right) -$$

$$= n\left[\frac{1}{n} - \frac{1}{n^2 2} + \frac{1}{n^3 3} - \cdots\right]$$

$$= 1$$

$$\left[\frac{1}{n} - \frac{1}{n^2 2} + \frac{1}{n^3 3} - \cdots\right] - \left[-\frac{1}{2n} + \frac{1}{3n^2} - \cdots\right]$$

$$\lim_{n\to\infty} n \log\left(\frac{u_n}{u_{n+1}}\right) = \frac{3}{2} > 1$$

$\therefore$ Log test $\sum u_n$ is cgt

if $x = 1/e$ ⇌

Qu: $1 + 2!\frac{x}{2} + \frac{2!}{3^2}x^2 + \frac{3!}{4^3}x^3 + \cdots$

$$u_n = \frac{(n+1)!\, x^{n+1}}{n^{n+1}}$$

$$u_{n+1} = \frac{(n)!\, x^n}{(n+1)^n}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)!\, x^{n+1}}{n^n} \times \frac{(n+1)^n}{n!\, x^n}$$

$$= \frac{x(1+\frac{1}{n})^n}{x}$$

$$\lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}\right) = \frac{e}{1}$$

if $\frac{e}{x} = 1$ test fail
at $x = e$.

$$\therefore \log \frac{u_n}{u_{n+1}} = \frac{(r + 1/n)^n}{e}$$

$$\log \frac{u_n}{u_{n+1}} = n \log\left(1 + \frac{1}{n}\right) - \log e$$

$$= n\left[\frac{1}{n} - \frac{1}{n^2 2} + \frac{1}{n^3 3} - \cdots\right]$$

$$\lim_{n \to \infty} \log\left(\frac{u_n}{u_{n+1}}\right) = \frac{1}{2} < 1$$

by log test Σu_n is cgt.

$\longrightarrow$ if $x = e$

Cauchy Root test.

If Σu_n is a positive term series such that

$$\lim_{n \to \infty} u_n (u_n)^{1/n} = K$$

then Σu_n is cgt if $K < 1$
Σu_n " " if $K > 1$

test fail if $K = 1$

when Apply ... more

Ex: $\sum \left(\frac{n}{n+1}\right)^{n^2}$

$$u_n = \left(\frac{n}{n+1}\right)^{n^2}$$

$$(u_n)^{1/n} = \left(\frac{n}{n+1}\right)^{n}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^{n}}$$

$$\lim_{n\to} (u_n)^{1/n} = \frac{1}{e} < 1$$

$\therefore$ by C.R.I $\sum u_n$ is cgt.

Q. $\sum \frac{(n+1)^n x^n}{n^n \cdot n}$

$\sum \frac{(n+1)x^n}{n^{n+1}}$

$$u_n = \frac{(n+1)^n x^n}{n^n \cdot n}$$

$$u_n^{1/n} = \frac{(n+1)(n)}{n(n)^{1/2}}$$

$$= \frac{(1+1/n)^n}{n^{1/n}}$$

$$\lim_{n\to\infty} u_n^{1/n} = x$$

x^x

if $x < 1 \Rightarrow \Sigma u_n$ is cgt

" $x > 1 \Rightarrow \Sigma u_n$ is dgt

if $x = 1$, Test fails

$$u_n = \frac{(n+1)^n}{n^{n+1}}$$

$$= \frac{n^n}{(n^{n+1})} \left(1 + \frac{1}{n}\right)^n$$

$$u_n = \frac{1}{n}$$

$\lim_{n \to \infty} \frac{u_n}{u_n} = e \neq 0$ & finite

$\therefore$ by p test $\Sigma u_n = \Sigma \frac{1}{n}$

$\therefore$ dgt $\because p = 1$

by Comp. test Σv_n is dgt if $x = 1$

Q. n^{th} term of Series.

$$\left[\frac{2^2}{1^2} - \frac{2}{1}\right]^{-1} + \left[\frac{3^3}{2^3} - \frac{3}{2}\right]^{-2} + \left[\frac{4^4}{3^4} - \frac{4}{3}\right]^{-3} - -$$

$$u_n = \left[\frac{(n+1)^{(n+1)}}{n^{(n+1)}} - \frac{(n+1)}{n}\right]^n$$ Here

(4) $u_n^{1/n} = \left[\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right]^{-1}$

$= \left[\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n - \left(1+\frac{1}{n}\right)\right]^{-1}$

$\lim_{n\to\infty} \lim u_n^{1/n} = [e-1]^{-1}$

$= \frac{1}{e-1} < 1$

$\therefore$ by CAT $\sum u_n$ is cgt.

Q.

$\frac{1}{9} + \frac{2x}{3} + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \cdots$

$n \cdot x^{n-1}$

$u_n = \left(\frac{n}{n+1}\right)^{n-1} x^{n-1}$

ewise 1 term

$u_n = \left(\frac{n+1}{n+2}\right)^n x^n$

$u_n^{1/n} = \left(\frac{n+1}{n+2}\right) x$

$\lim_{n\to\infty} u_n^{1/n} = x$

if $x < 1 \Rightarrow \sum u_n$ is cgt

$x > 1 \Rightarrow$ " dgt

if $x = 1$ test fail.

$$u_n = \left(\frac{n+1}{n+2}\right)^n$$

$$= \frac{(1+1/n)^n}{(1+\frac{2}{n})^n}$$

$$\lim_{n\to\infty} u_n = \frac{e}{e^2} = \frac{1}{e} \neq 0.$$

$\therefore$ by C.F.T Σu_n is dgt if $x=1$.

Q. $\Sigma u_n = \Sigma \frac{n^3}{3^n}$

$$u_n = \frac{n^3}{3^n}, \quad u_{n+1} = \frac{(n+1)^3}{3^{n+1}}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^3}{3^n} \times \frac{3^{n+1}}{(n+1)^3}$$

$$= \frac{3}{(1+\frac{1}{n})^3}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 3 > 1$$

$\therefore$ Cgt.

Q. $\Sigma 3^{-n-(-1)^n}$

$$u_n = 3^{-n+1} \quad \text{if } n \text{ is even}$$

$$u_n = 3^{-n+1} \quad \text{if } n \text{ is odd}$$

$\frac{1}{\log \infty} = 0$

$u_n = 3^{-n} \cdot \frac{1}{3}$ — even

$u_n = 3^{-3} \cdot 3$ — odd

$u_n^{1/n} = 3^{-1} \cdot \left(\frac{1}{3}\right)^{1/n}$ — even

$= 3^{-1} \cdot 3^{1/n}$ — odd

$\lim_{n \to \infty} u_n = \frac{1}{3}$ → even

$\lim_{n \to \infty} u_n = \frac{1}{3}$ → odd

< 1

by C.R.I $\sum u_n$ is cgt

Q. $\sum \frac{1}{(\log n)^n}$

$u_n = \frac{1}{(\log n)^n}$

$u_n^{1/n} = \frac{1}{\log n}$

$\lim_{n \to \infty} u_n^{1/n} = 0 < 1$

$\Rightarrow$ cgt.

* Cauchy Integral test :-

A positive term series

$$f(1) + f(2) + f(3) - - - f(n) - - -$$

where $f(n)$ decreases as an increases converges or diverges according to the integral

$$\int_1^{\infty} f(x)\,dx \text{ is finite or infinite.}$$

Q. $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ decreasing Fn.

$$f(n) = \frac{1}{n \log n}$$

$$\int_2^{\infty} \frac{1}{n \log n}\,dn$$

Let $\log x = t \qquad \frac{1}{n}\,dx = dt$

$$\int_{\log 2}^{\infty} \frac{dt}{t}$$

$$= [\log \infty - \log(\log 2)]$$

$$= \infty$$

Hence By CIT $\sum u_n$ is dgt

Q. $\sum_{n=1}^{\infty} n e^{-n^2}$

$$\int_1^{\infty} n e^{-n^2}$$

$$n^2 = t \Rightarrow 2n\,dn = dt$$

$$\int_1^{\infty} e^{-t} \frac{dt}{2}$$

$$\frac{1}{2}\left[-e^{-t}\right]_1^{\infty}$$

$$\frac{1}{2}\left[0 + e^{-1}\right) = \frac{1}{2e} =$$

by CIT $\sum u_n$ is cgt. $\Rightarrow$

Q $\sum \frac{1}{\sqrt{n}}$ dgt.

Q. $\sum_1^{\infty} \frac{1}{n^2+1}$ Comp. test, Convergent

$$\int_1^{\infty} \frac{1}{n^2+1}\,dx = \left[\tan^{-1} x\right]_1^{\infty}$$

$$\left[\tan^{-1}\infty - \tan^{-1} 1\right] = \frac{\pi}{2} - \frac{\pi}{4} = \text{Q}$$

never be divergent series

not conv. then Σu_n is oscillatory

X Alternating series

The series with alternate sign is called alternate series

$(-1)^n = \Sigma u_n$ (denote).

Sum = Infinite / -Infinite

Two type of alternating series.

- Absolutly Convergent series
- Conditionally cgt series.

Absolute cgt series:

the alternate series is

$|u_n|$ = conv.

absoulete conv

4. $(-1)^n \Sigma u_n$ is absolutely cgt

If $|u_n| + |-u_2| + \cdots$ is convergent

IF the given series is absolutely cgt then the alternating series is also cgt. by ~~conv~~ if the given alt. series is not absolutely convergent the the alternating series may be cgt may not be cgt.

* Leibinitz's test :
An imp. series $\sum (-1)^{n-1} u_n$ is Cgt

1. Cgt if $u_{n+1} \leq u_n \Rightarrow \boxed{u_n - u_{n+1} \geq 0}$

2. $\lim_{n \to \infty} u_n = 0$

For Ex: show that the series is converges

$1 - \frac{1}{2} + \frac{1}{3} + \cdots$

$$\sum (-1)^{n-1} \frac{1}{n}$$

$$u_n = \frac{1}{n} \quad , \quad u_{n+1} = \frac{1}{n+1}$$

$$u_n - u_{n+1} = \frac{1}{n} - \frac{1}{n+1}$$

$$= \frac{1}{n(n+1)} \geq 0$$

2) $\lim_{n \to \infty} u_n = 0$

Q Show that the exponential series

$$\sum (-1)^{n+1} (\sqrt{n+1} - \sqrt{n})$$

is semi-convergent or conditionally Cgt

$$u_{n+1} = (\sqrt{n+1} - \sqrt{n})$$

$$|u_n| = \frac{\sqrt{(n+1)} - \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \times \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} =$$

$$= \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$u_n = \frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{1+\frac{1}{n}} + 1} \right)$$

$$v_n = \frac{1}{\sqrt{n}}$$

$$\frac{u_n}{v_n} = \frac{1}{\sqrt{1+\frac{1}{n}} + 1}$$

$$\lim_{n \to \infty} \frac{u_n}{v_n} = \frac{1}{2} \neq 0 \text{ \& finite}$$

$\therefore$ by P test $\sum v_n = \frac{1}{\sqrt{n}}$ is dgt because $P = \frac{1}{2} < 1$

$\therefore$ by Comparison test $|u_n|$ is dgt.

$$u_n = \sqrt{n+1} - \sqrt{n}$$

$$u_{n+1} = \sqrt{n+2} - \sqrt{n+1}$$

$$u_n \geq u_{n+1}$$

$u_n - u_{n+1} = \sqrt{n+1} - \sqrt{n} - \sqrt{n+2} + \sqrt{n+1}$

$= 2\sqrt{n+1} - \sqrt{n} - \sqrt{n+2} \geq 0$

which satisfy the (I) condition

$$\lim_{n\to\infty} u_n = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$= 0$

therefore, both condition are satisfy.

So, by it is cgt.

by beibinitz test it is cgt

Que: $\frac{2}{1^2} - \frac{3}{2^2} + \frac{4}{3^2} - \frac{5}{4^2} - \cdots$

$\sum (-1)^{(n+1)} \frac{n+1}{n^2} \Rightarrow |u_n| = \frac{(n+1)}{n^2}$

$u_n = \frac{1}{n}\left(1 + \frac{1}{n}\right)$

$v_n = \frac{1}{n}$

$\frac{u_n}{v_n} = \left(1 + \frac{1}{n}\right)$

$\lim_{n\to\infty} \frac{u_n}{v_n} = 1 \neq 0$ & finite

∴ therefore by P test $\sum u_n = \sum \frac{1}{n}$ is dgt ∵ $P = 1$

∴ by Comp. test $\Sigma |u_n|$ is dgt.

$$\Sigma (-1)^{n+1} \frac{(n+1)}{n^2}$$

$$u_n - u_{n+1} = \frac{n+1}{n^2} - \frac{(n+2}{(n+1)^2}$$

$$= \frac{(n+1)^3 - (n^3+2n^2)}{(n+1)^2 n^2}$$

$$= \frac{(1+3n^2+3n) - 2n^2}{(n+1)^2 n^2}$$

$$= \frac{1+n^2+3n}{(n+1)^2 n^2} \geq 0$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right) = 0$$

Que: Σ (1) $(-1)^{n+1} \frac{n^2}{(n+1)!}$ I

$$u_n = \frac{n^2}{n+1}$$

Using de-Alambert test