Unit - 3

25/09/18

* Fourier Series :-

If $f(x)$ is Periodic function with Period $2l$

i.e $f(x+2l) = f(x)$

in the interval $(-l, l)$ then the fourier series of the function $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \rightarrow (1)$$

Even function

$$f(-x) = f(x)$$

$$\int_{-l}^{l} f(x)\,dx = 2\int_{0}^{l} f(x)\,dx$$

Odd function: $f(-x) = -f(x)$.

$$\int_{-l}^{l} f(x)\,dx = 0$$

Results :-

(1) $$\int_{-\pi}^{\pi} \sin nx\,dx = 0$$

3, 4, 5 → 2 Que.

(2) $\int_{-\pi}^{\pi} \cos nx\,dx = 2\int_{0}^{\pi} \cos \pi x\,dx = 2\left[\frac{\sin nx}{n}\right]_0^{\pi} = 0$

(3) $\int_{-\pi}^{\pi} \cos mx \cos nx\,dx =$

$= 2\int_{0}^{\pi} \cos(m+n)x + \cos(m-n)x\,dx$

$= 0$

(4) $\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = 2\int_{0}^{\pi} \sin mx \sin nx\,dx$

$2\int_{0}^{\pi} [\cos(m+n)x - \cos(m-n)x]\,dx$

$= 0$

(5) $\int_{-\pi}^{\pi} \cos^2 nx = 2\int_{0}^{\pi} \left(\frac{1+\cos 2mx}{2}\right)dx$

$= \left[x + \frac{\sin(2nx)}{2n}\right]_0^{\pi} = \pi$

Cons. → always even fn.

(6) $$\int_{-\pi}^{\pi} \sin^2 nx \, dx = 2\int_{0}^{\pi} \sin^2 nx$$

$$= \int_{0}^{\pi} (1 - \cos 2nx) \, dx$$

$$= \pi$$

Try

(7) $$\int e^{ax} \sin bx \, dx =$$

$$= \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b \cos bx\right]$$

Try.

(8) $$\int e^{ax} \cos bx \, dx$$

$$= \frac{e^{ax}}{a^2+b^2}\left[a \cos bx + b \sin bx\right]$$

Integrating eq^n (1) from limit $(-\pi, \pi)$

$$\int_{-\pi}^{\pi} f(x) \, dx = \frac{a_0}{2}\int_{-\pi}^{\pi} dx + a_n \int_{-\pi}^{\pi} \cos nx \, dx$$

$$= \int_{-\pi}^{\pi} f(x)\,dx = a_0[\pi]$$

$$\boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx}$$

multiplying both sides by $\cos nx$ in eq^n (1)

and integrate from $(-\pi, \pi)$ w.r.t x.

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx =$$

$$= \frac{a_0}{2}\int_{-\pi}^{\pi} \cos nx\,dx + a_n\int_{-\pi}^{\pi} \cos^2 nx\,dx$$

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx = a_n\pi$$

$$\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx}$$

multiply both sides by $\sin nx$ in eq^n (1)

and integrate w.r.t $(-\pi, \pi)$

$$\int_{-\pi}^{\pi} f(x) \sin nx \, dx = \int_{-\pi}^{\pi} b_n \sin^2 nx \, dx$$

$$= \pi b_n$$

$$\boxed{b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx}$$

* If the f^n is defin $0 < x < 2c$

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c}}$$

$$a_0 = \frac{1}{c} \int_0^{2c} f(x) \, dx$$

$$a_n = \frac{1}{c} \int_0^{2c} f(x) \cos \frac{n\pi x}{c} \, dx$$

$$b_n = \frac{1}{c} \int_0^{2c} f(x) \sin \frac{n\pi x}{c} \, dx$$

Example :-

Obtain the fourier series

$$f(x) = \frac{1}{4}(\pi - x)^2 \quad \text{in} \quad 0 < x < 2\pi$$

Hence obtain the following relation.

$$\frac{1}{1^2} + \frac{1}{2^2} + - - - - = \frac{\pi^2}{6}$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - - - - = \frac{\pi^2}{12}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} - - - - = \frac{\pi^2}{8}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} (\pi - x)^2 \, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} \left[\frac{\pi^2}{4} + \frac{x^2}{4} - \frac{\pi x}{2} \right] dx$$

$$= \frac{1}{\pi} \left[\frac{\pi^2}{4} x + \frac{x^3}{12} - \frac{\pi x^2}{4} \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\frac{2\pi^3}{4_2} + \frac{8\pi^3}{12_3} - \frac{4\pi^3}{4} \right]_0^{2\pi}$$

$$= \frac{\pi^2}{6}$$

(2) diff. zero

(2) (1)

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi-x)^2}{4}\cos nx\,dx$$

$$= \frac{1}{4\pi}\left[\frac{\sin nx}{n}(\pi-x)^2 - \int_0^{2\pi} -2(\pi-x)\frac{\sin nx}{n}dx\right]$$

$$= \frac{2}{4n\pi}\left[\int_0^{2\pi}(\pi-x)\sin nx\,dx\right]$$

$$= \frac{1}{2n\pi}\left[-\frac{\cos nx}{n}(\pi-x) - \int_0^{2\pi}(-1)\left(\frac{-\cos nx}{n}\right)\right]$$

$$= -\frac{1}{2n^2\pi}\left[\cos nx\,(\pi-x)\right]_0^{2\pi}$$

$$= -\frac{1}{2n^2\pi}\left[\cos 2n\pi\,(\pi-2\pi) - \pi\right]$$

$$= -\frac{1}{2n^2\pi}\left[-\pi-\pi\right]$$

$$a_n = \frac{1}{n^2}$$ [which is valid for all value of n]

By Part integral :

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi-x)^2}{4} \sin nx\, dx$$

$$= \frac{1}{4\pi}\left[-\frac{\cos nx}{n}(\pi-x)^2 - \int_0^{2\pi} 2(\pi-x)\frac{\cos nx}{n}\, dx \right]$$

$$= \frac{1}{4\pi}\left[\left(-\frac{\pi^2}{n} + \frac{\pi^2}{n} \right) - \frac{2}{n}\left[(\pi-x)\frac{\sin nx}{n} - \int_0^{2\pi} (-1)\frac{\sin nx}{n} dx \right] \right]$$

$$= \frac{1}{4\pi}\left[\frac{-2}{n}\left[-\frac{\cos nx}{n^2} \right]_0^{2\pi} \right]$$

$$= \frac{1}{2n^3\pi}\left[\cos nx \right]_0^{2\pi}$$

$$= \frac{1}{2n^3\pi}[1-1] = 0$$

$$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}\cos nx$$

$$= \frac{\pi^2}{12} + \left[\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x \cdots \right]$$

$$\frac{(\pi-x)^2}{4} =$$

at $x = 0$

$$\frac{\pi^2}{4} = \frac{\pi^2}{12}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{\pi^2}{6} = \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] \quad (A)$$

at $x = \pi$

$$0 = \frac{\pi^2}{12} + \left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} - \cdots\right]$$

$$-\frac{\pi^2}{12} = \left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right]$$

$$\frac{\pi^2}{12} = \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] \quad (B)$$

adding (A) and B

$$\frac{\pi^2}{4} = 2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\frac{\pi^2}{8} = \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

even $b_n = 0$ even

zero $a_n = a_0 = 0$ odd.

Que. $f(x) = x + x^2 \quad -\pi < x < \pi$

find the fourier seri

Hence reduce $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Rmk:

Note: If the given f^n $f(x)$ is even function and the limit of the f^n values $(-l, +l)$ in that case b_n is always zero

on the other side if $f(x)$ is odd f^n and f^n is define from $(-l, l)$ in that case a_0 and a_n must be zero, only b_n will exist.

All f^n will not zero

$$\rightarrow a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{\pi} x\, dx + \int_{-\pi}^{\pi} x^2 dx \right]$$

~~CLASS~~ $\cos n\pi = (-1)^n$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 dx$$

$$= \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi} \times 2\left[\int_0^{\pi} x^2 \cos nx\, dx\right]$$

$$= \frac{2}{\pi}\left[x^2 \frac{\sin nx}{n}\Big|_0^{\pi} - \int_0^{\pi} 2x \frac{\sin nx}{n}\, dx\right]$$

$$= -\frac{4}{n\pi}\int_0^{\pi} x \sin nx\, dx$$

$$= \frac{-4}{n\pi}\left[-x\frac{\cos nx}{n} - \int -\frac{\cos nx}{n}\, dx\right]$$

$$= \frac{-4}{n\pi}\left[-\pi\frac{\cos n\pi}{n} + \left[\frac{\sin nx}{n^2}\right]_0^{\pi}\right]$$

$$= +\frac{4}{n^2}\left[(-1)^n\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \int_0^{\pi} 1\cdot\left(\frac{-\cos nx}{n}\right)dx\right]$$

$$= \frac{2}{\pi}\left[-\pi\frac{\cos n\pi}{n} + \left[\frac{\sin nx}{n^2}\right]_0^{\pi}\right]$$

$$b_n = -\frac{2}{n}(-1)^n$$

$$= \frac{2\pi^2}{3\times 2} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n \times\cos nx + \sum_{n=1}^{\infty}\left(\frac{-2}{n}(-1)^n\right)\times\sin nx$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx - \sum_{n=0}^{\infty}\frac{2}{n}(-1)^n\sin nx$$

~~$= \frac{\pi^2}{3} + \left(\frac{-4}{n^2}\cos nx\right) - \left(\frac{-2}{n}\right)\sin$~~

$$f(x) = \frac{\pi^2}{3} + 4\left[\frac{-1}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \cdots\right]$$

$$-\frac{2}{n}\left[\frac{-1}{1}\sin x + \frac{1}{2}\sin 2x + \cdots\right]$$

SBG STUDY

$f(x) = f(-\pi)$

at $x = -\pi$

$$\lim_{h \to 0} \frac{f(-\pi - h) + f(-\pi + h)}{2}$$

$$\frac{f(-\pi - 0) + f(-\pi + 0)}{2}$$

$$\frac{f(\pi - 0) + f(-\pi + 0)}{2}$$

$$\frac{f(\pi + \pi^2) + f(-\pi + (-\pi)^2}{2}$$

$$f(-\pi) = \pi^2$$

$$\pi^2 = \frac{\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \cdots\right]$$

$$\frac{2\pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} - \cdots\right]$$

$$\frac{\pi^2}{6} = \left[\frac{1}{1^2} + \frac{1}{2^2} + \cdots\right]$$

Imp. (must solve)

Que: $f(x) = x \sin x$ (i) $-\pi < x < \pi$ (ii) $0 < x < 2\pi$

(90%)

Que: $f(x) = e^x$ $-\pi < x < \pi$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x dx$$

$$= \frac{1}{\pi}\left[e^{x}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[e^{\pi} - e^{-\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{e^{\pi} - e^{-\pi}}{2}\right]$$

$$a_0 = \frac{2\sinh\pi}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{x}\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{e^{x}}{1+n^2}\left[\cos nx + n\sin nx\right]\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{\pi}\cos n\pi - e^{-\pi}\cos n\pi\right]$$

$$= \frac{2(-1)^n}{\pi(1+n^2)}\left[\frac{e^{\pi} - e^{-\pi}}{2}\right]$$

$$a_n = \frac{2(-1)^n \sinh\pi}{\pi(1+n^2)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}\left[\sin nx - n\cos nx\right]\right]_{-\pi}^{\pi}$$

$$= \frac{n}{\pi(1+n^2)}\left[-e^{\pi}\cos n\pi + e^{-\pi}\cos n\pi\right]$$

$$= \frac{-2n(-1)^n}{\pi(1+n^2)}\left[\frac{e^{\pi}-e^{-\pi}}{2}\right]$$

$$= \frac{-2n(-1)^n}{\pi(1+n^2)}\sinh\pi .$$

$$f(x) = \frac{\sinh x}{\pi} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n \sinh\pi}{\pi(1+n^2)}\right]\cos nx + \sum_{n=1}^{\infty}\left[\frac{-2n(-1)^n \sinh\pi}{\pi(1+n^2)}\right]\sin nx$$

**

Que: $f(x) = \left(\frac{\pi - x}{2}\right)^2 \qquad 0 < x < 2\pi$

Q. $f(x) = x \sin x \quad -\pi < x < \pi$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x \sin x \, dx$$

$$= \frac{2}{\pi}\left[-x\cos x + \sin x\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\pi\cos\pi\right]$$

$$= 2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= \frac{1}{\pi}\left[x.\left(\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right)\right]$$

$$- \int_0^{\pi}\left(\frac{-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)x}{n-1}\right)dx$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{-\cos(n+1)\pi}{n+1}\right) + \frac{\cos(n-1)\pi}{n-1}\right]$$

$$\left[\frac{(-1)^{n+2}}{n+1} + \frac{(-1)^{n+1}}{n-1}\right]$$

$$(-1)^{n+1}\left[\frac{-1}{n+1} + \frac{1}{n-1}\right]$$

$$a_n = (-1)^{n+1}\left[\frac{2}{n^2-1}\right] \qquad n \neq 1$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos n\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \sin 2x\, dx$$

$$= \frac{1}{\pi}\left[\frac{-x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{\pi}$$

$$a_1 = \frac{1}{\pi}\left[\frac{-\pi}{2}\right) = \frac{-1}{2}$$

$$b_n = 0$$

$$f(n) = 1 + \left(\frac{-1}{2}\right)$$

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2} a_n \cos nx$$

$$f(x) = 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} (-1)^{n-1} \left(\frac{2}{n^2-1}\right) \cos nx$$

Q. Find the Fourier Series of the f

$$f(x) = -1 \qquad -\pi < x < -\frac{\pi}{2}$$

$$= 0 \qquad -\pi/2 < x < \pi/2$$

$$= 1 \qquad \pi/2 < x < \pi$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} f(x)dx + \int_{-\pi/2}^{\pi/2} f(x)dx + \int_{\pi/2}^{\pi} f(x)dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} -dx + \int_{\pi/2}^{\pi} dx\right]$$

$$= \frac{1}{\pi}\left[-\left(-\frac{\pi}{2} + \pi\right) + \left[\pi - \pi/2\right]\right]$$

$$a_0 = \frac{1}{\pi}(0) = 0$$

$\cos\frac{n\pi}{2}$ = ddd = 0, even = 0 (−1, 1)

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} -\cos nx\,dx + \int_{\pi/2}^{\pi} \cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left[\frac{-\sin nx}{n}\right]_{-\pi}^{-\pi/2} + \left[\frac{\sin nx}{n}\right]_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left(\frac{\sin n\pi/2}{n}\right) + \left(\frac{-\sin n\pi/2}{n}\right.$$

$$= 0$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} -\sin nx\,dx + \int_{\pi/2}^{\pi} \sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left[\frac{\cos nx}{n}\right]_{-\pi}^{-\pi/2} + \left[\frac{-\cos nx}{n}\right]_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi/2}{n} - \frac{\cos n\pi}{n}\right] + \left(\frac{-\cos^n\pi}{n} + \frac{\cos n\pi/2}{n}\right)$$

$$= \frac{2}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right]$$

$$b_n = \frac{2}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right]$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(\cos\frac{n\pi}{2} - \cos n\pi\right)\sin nx$$

Q.

$$f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

Reduce

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots = \frac{\pi^2}{8}$$

$f(x) =$

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\, dx + \int_{0}^{\pi} x\, dx\right]$$

$$= \frac{1}{\pi}\left[\left[-\pi x\right]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\pi(\pi) + \frac{\pi^2}{2}\right) \quad \text{(1)}$$

$$= -\pi/2$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-n}^{0} + \left(\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right)_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$a_n = 0$ if n is even

$\frac{-2}{n^2\pi}$ if odd.

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left(\pi\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left[\frac{-x\cos nx}{n} + \right.$$

$$\left. + \frac{\sin nx}{n^2}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{1}{n} - \frac{(-1)^n}{n}\right) + \left(-\pi\frac{(-1)^n}{n}\right)\right]$$

Periodic

$f(\pi) = f(\pi - 2\pi)$

$$= \left[\frac{1}{n} - \frac{(-1)^n}{n} - \frac{(-1)^n}{n}\right]$$

$$b_n = \left[\frac{1}{n} - \frac{2(-1)^n}{n}\right]$$

$$f(x) = \frac{-\pi}{2 \times 2} + \left[\frac{-2}{1^2\pi}\cos x - \frac{2}{3^2\pi}\cos 3x \ldots\right] =$$

$$+ \left[\frac{3}{1}\sin x - \frac{1}{2}\sin 2x \ldots\right]$$

$$= \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$$

at $x = \pi$. discontinious fn.

$$f(\pi) = \frac{f(\pi - 0) + f(\pi + 0)}{2}$$

$$= \frac{\pi + f(\pi - 2\pi + 0)}{2}$$

$$= \frac{\pi + f(-\pi)}{2}$$

$$= \frac{\pi - \pi}{2} = 0$$

$$0 = -\frac{\pi}{4} + \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{\pi}{4} = \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\frac{\pi^2}{8} = \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

= DIY

Qu 1: (1) $f(x) = x + \pi \qquad 0 \le x \le \pi$

$= -x - \pi \qquad -\pi \le x < 0$

Imp. (9)

$f(x) = x \qquad -\pi < x < 0$

$= -x \qquad 0 < x < \pi$

Hence Reduce

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots = \frac{\pi^2}{8}$$

Sol^{n} (1)

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x-\pi)\,dx + \int_{0}^{\pi}(x+\pi)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left[-\frac{x^2}{2} - \pi x\right]_{-\pi}^{0} + \left[\frac{x^2}{2} + \pi x\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi^2}{2} - \pi^2\right) + \left(\frac{\pi^2}{2} + \pi^2\right)\right]$$

$$= \pi$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x-\pi)\cos nx\,dx + \int_{0}^{\pi}(x+\pi)\cos nx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{-\sin nx}{n}(+x+\pi) - \frac{\cos nx}{n^2}\right)_{-\pi}^{0} + \left((x+\pi)\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right)_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left[\left(\frac{-1}{n^2} + \frac{(-1)^n}{n^2}\right) + \left(\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right)\right]\right]$$

$$= \frac{2}{n^2\pi}$$

$0 =$ even

$\frac{-4}{n^2\pi} =$ odd.

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x-\pi)\sin nx\,dx + \int_{0}^{\pi}(x+\pi)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left((x+\pi)\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right)_{-\pi}^{0} + \left(-(x+\pi)\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right)_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n} + \frac{(-2\pi)(-1)^n}{n} + \frac{\pi}{n}\right]$$

$$= \frac{2}{n}\left[1-(-1)^n\right]$$

$$= 0 \quad \text{— even}$$

$$= \frac{4}{n} \quad \text{— odd.}$$

Qu: $f(x) = \begin{cases} 2 & -2 \le x \le 0 \\ x & 0 < x < 2 \end{cases}$

$2c = 4$

$c = 2$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}$$

Learn

$$a_0 = \frac{1}{c}\left[\int_{-2}^{2} f(x)\,dx + \int_{0}^{2} f(x)\,dx\right]$$

$$= \frac{1}{2}\left[\int_{-2}^{0} 2\,dx + \int_{0}^{2} x\,dx\right]$$

$$= \frac{1}{2}\left[\left[2x\right]_{-2}^{0} + \left[\frac{x^2}{2}\right]_{0}^{2}\right]$$

$$= \frac{1}{2}\left(+4 + \frac{4}{2}\right) = \frac{1}{2} \times 6 = 3$$

$$a_n = \frac{1}{2}\left[\int_{-2}^{0} 2\cos\frac{n\pi x}{2}\,dx + \int_{0}^{2} x\cos\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[\left[\frac{4}{n\pi}\sin\frac{n\pi x}{2}\right]_{-2}^{0} + \left[\frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right]_{0}^{2}\right]$$

don't come infinite — in constant value.

$$= \frac{1}{2}\left[\frac{4}{n^2\pi^2}(\cos n\pi - 1)\right]$$

$$= \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$= 0$ if n is even

$= \frac{-4}{n^2\pi^2}$, if n is odd.

$$b_n = \frac{1}{2}\left[\int_{-2}^{0} 2\sin\frac{n\pi x}{2}\,dx + \int_{0}^{2} x\sin\frac{n\pi}{2}x\,dx\right]$$

$$= \frac{1}{2}\left[\left(\frac{-4}{n\pi}\cos\frac{n\pi x}{2}\right)_{-2}^{0} + \left(\frac{-2x}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)_{0}^{2}\right]$$

$$= \frac{1}{2}\left[\frac{-4}{n\pi}\left(1-(-1)^n\right) + \left(\frac{-4}{n\pi}(-1)^n\right)\right]$$

$$b_n = \frac{-2}{n\pi}$$

$$f(x) = \frac{3}{2} + \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{-4}{n^2\pi^2}\cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty} \frac{-2}{n\pi}\sin\frac{n\pi x}{2}$$

Que: $f(x) = |\cos x| \quad -\pi < x < \pi$

Que: $4 - x^2 \quad 0 < x < 2$

$c = 1$

$$a_0 = 1 \int_0^2 (4 - x^2)\, dx$$

$$= \left[4x - \frac{x^3}{3}\right]_0^2$$

$$= 8 - \frac{8}{3} = \frac{16}{3}$$

$$a_n = \int_0^2 (4 - x^2) \cos \frac{n\pi x}{1}\, dx$$

$$= \left[(4 - x^2)\frac{\sin n\pi x}{n\pi} + \int_0^2 2x \frac{\sin n\pi x}{n\pi}\, dx\right]$$

$$a_n = \frac{2}{n\pi}\left[-x\frac{\cos n\pi x}{n\pi} + \frac{\sin \pi x}{n^2\pi^2}\right]_0^2$$

$$= \frac{+2}{n\pi}\left(\frac{-2}{n\pi}\right)$$

$$= -\frac{4}{n^2\pi^2}$$

$$b_n = \int_0^2 (4 - x^2) \sin n\pi x \, dx$$

$$= \left[(4 - x^2) \frac{\cos n\pi x}{n\pi} + \int 2x \frac{\cos n\pi x}{n\pi} dx \right]$$

$$= \left[\frac{4}{n\pi} - \frac{2}{n\pi} \left[x \frac{\sin n\pi x}{n\pi} + \frac{\cos n\pi x}{n^2\pi^2} \right] \right]_0^2$$

$$= \frac{4}{n\pi} - \frac{2}{n\pi} \left[\frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2} \right]$$

$$b_n = \frac{4}{n\pi}$$

$$f(x) = \frac{8}{3} + \sum_{n=1}^{\infty} \frac{-4}{n^2\pi^2} \cos n\pi x + \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin n\pi x$$

at discontinuities
$= \frac{f(x-) + f(x+)}{2}$

* Half Range Series :

(1) Half Range Cosine Series :

Some times it is required to expand a fn $f(x)$ in the range $(0, l)$ if we need half range cosine series. In that case b_n must be zero

so, we will assume a fn $f(x)$ in the range $(-l, 0)$ such that the fn $f(x)$ is even function. in the range $(-l, l)$.

and there for the half range cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\, dx$$

$$\boxed{a_0 = \frac{2}{l} \int_{0}^{l} f(x)\, dx}$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l}\, dx$$

$$\boxed{= \frac{2}{l} \int_{0}^{l} f(x) \cos \frac{n\pi x}{l}\, dx}$$

3 Question

If we need half Range Sine series then a_0 and a_n should be zero

for that we will assume the f^n $f(x)$ from $(-l, 0)$ is such that the f^n $f(x)$ is odd f^n in the range $(-l, l)$

and therefore the half range Sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \, dx$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \, dx$$

$$\boxed{b_n = \frac{2}{l} \int_{0}^{l} f(x) \sin \frac{n\pi x}{l} \, dx}$$

Q. Expend $(\pi x - x^2)$ in a half range sine series in the interval $(0, \pi)$. upto three terms [because half range series a_0 and $a_0 = 0$]

$l = \pi$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty} b_n \sin nx \, dx.$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\sin nx\,dx.$$

$$= \frac{2}{\pi}\left[-(\pi x - x^2)\frac{\cos nx}{n} - \frac{1}{n}\int (\pi - 2x)\cos nx\,dx\right]$$

$$= \frac{2}{n\pi}\left[(\pi - 2x)\frac{\sin nx}{n} - \frac{2\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{-4}{n^3\pi}(\cos n\pi - 1)$$

$$= \frac{-4}{n^3\pi}((-1)^n - 1)$$

$b_n = 0$ if n is even

$$= \frac{8}{n^3\pi} \quad \text{odd.}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{8}{n^3\pi}\sin nx$$

n odd

$$= \frac{8}{\pi}\left[\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} \ldots\right)$$

Q. $f(x) = x^2$ $\quad l = 3$

$0 < x < 3$

Find half range cosine series

half range cosine series.

$b_n = 0$ $\quad \boxed{l = 3}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{3}$$

$$a_0 = \frac{2}{3} \int_0^3 x^2 \, dx$$

$$= \frac{2}{3} \left[\frac{x^3}{3} \right]_0^3 = \frac{2}{3} \left[\frac{27}{3} \right] = 6.$$

$$a_n = \frac{2}{3} \int_0^3 x^2 \cdot \cos \frac{n\pi x}{3} \, dx$$

$$= \frac{2}{3} \left[\frac{3x^2}{n\pi} \cdot \sin \frac{n\pi x}{3} - \int \frac{2x \times 3}{n\pi} \sin \frac{n\pi x}{3} dx \right]$$

$$= \frac{2}{3} - \frac{4}{n\pi} \int x \sin \frac{n\pi x}{3} \, dx$$

$$= \frac{-4}{n\pi} \left[\frac{3x}{n\pi} \left(-\cos \frac{n\pi x}{3} \right) + \frac{9}{n^2\pi^2} \sin \frac{n\pi x}{3} \right]_0^3$$

$= \infty$ not one at any value

$$= \frac{12}{n^2\pi^2}\left[3\cos n\pi\right]$$

$$a_n = \frac{36(-1)^n}{n^2\pi^2}$$

$$f(x) = 3 + \sum_{n=1}^{\infty} \frac{36(-1)^n}{n^2\pi^2} \cos\frac{n\pi x}{3}$$

Que: $f(x) = |x|$ $\quad (-2, 2)$

find fourier series $\quad 2l = 4$, $l = 2$

$f(x)$ = even fn.

$b_n = 0$

$$a_0 = \frac{1}{2}\int_{-2}^{2} |x|\,dx$$

$$= \int_0^2 |x|\,dx = \left[\frac{x^2}{2}\right]_0^2 = 2$$

$$a_n = \frac{1}{2}\int_{-2}^{2}$$

$$f(x) = |x|$$
$$= -x \qquad -2 < x < 0$$
$$= x \qquad 0 < x < 2$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0} (-x)\,dx + \int_{0}^{2} (x)\,dx\right]$$

$$= \frac{1}{2}\left[\left[-\frac{x^2}{2}\right]_{-2}^{0} + \left[\frac{x^2}{2}\right]_{0}^{2}\right]$$

$$= \frac{1}{2}\left[+\frac{4}{2} + \frac{4}{2}\right] = \frac{1}{2} \times \frac{8}{2} = 2$$

$$a_0 = 2$$

$$a_n = \frac{1}{2}\left[\int_{-2}^{2} f(x) \cos\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[\int_{-2}^{0} (-x)\cos\frac{n\pi x}{2}\,dx + \int_{0}^{2} x \cos\frac{n\pi x}{2}\,dx\right]$$

$$\frac{1}{2}\left[\int_{-2}^{0}(-x)\cos\frac{n\pi x}{2}\,dx+\int_{0}^{2}x\cos\frac{n\pi x}{2}\,dx\right]$$

$$\frac{1}{2}\left[\right.$$

$$L=2$$

$$\frac{1}{2}\left[\left(-\left(\frac{2x}{n\pi}\sin\frac{n\pi x}{2}+\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right)_{-2}^{0}\right.$$

$$\left.+\left[\frac{+2x}{n\pi}\sin\frac{n\pi x}{2}+\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right]_{0}^{2}\right]$$

$$\frac{1}{2}\left[\left(\frac{-4}{n^2\pi^2}+\frac{4}{n^2\pi^2}(-1)^n\right)+\left(\frac{4}{n^2\pi^2}(-1)^n-\frac{4}{n^2\pi^2}\right)\right]$$

$$\left[\frac{-4}{n^2\pi^2}+\frac{4}{n^2\pi^2}(-1)^n\right]$$

$$\frac{4}{n^2\pi^2}\left[-1+(-1)^n\right]$$

$a_n = 0$ if even

$= \frac{-8}{n^2\pi^2}$ odd

$b_n = 0$ (f^4 is even fn)